Non-linear distributed parameter system estimation using two dimension Haar functions

Joon-Hoon Park, *Member, KIMICS*, **T.S. Sidhu**, *Nonmember*

Abstract—A method using two dimension Haar functions approximation for solving the problem of a partial differential equation and estimating the parameters of a non-linear distributed parameter system (DPS) is presented. The applications of orthogonal functions, including Haar functions, and their transforms have been given much attention in system control and communication engineering field since 1970's. The Haar functions set forms a complete set of orthogonal rectangular functions similar in several respects to the Walsh functions. The algorithm adopted in this paper is that of estimating the parameters of non-linear DPS by converting and transforming a partial differential equation into a simple algebraic equation. Two dimension Haar functions approximation method is introduced newly to represent and solve a partial differential equation.

The proposed method is supported by numerical examples for demonstration the fast, convenient capabilities of the method.

Index Terms—estimation of non-linear DPS, Haar transform, operation matrix, two dimension Haar functions approximation.

I. INTRODUCTION

Since 1970s, orthogonal transforms and their applications, such as Walsh, block pulse and Haar have been developed and used for solving analysis and optimization problems of dynamic systems. The basic theory of orthogonal transforms is to convert a ordinary and partial differential equation into an algebraic equation and operation matrices of integration are applied to simplify the problems. Parasekevopoulos and Bounas[1] propose a method of identifying the parameters of linear time invariant DPS using Walsh functions. But application of this method is limited to a first-order partial differential equation. Tzafestas and Stavroulakis[2] introduce Walsh and block pulse operational matrices for distributed parameter and delay systems. Sinha and Rajamani[3] apply the double Walsh series for estimating the parameters of non-linear DPS.

In this paper the problem of non-linear DPS estimation via Haar transform is considered and the two dimension Haar functions approximation concept for solving the problem of partial differential equation is introduced. Appropriate numerical examples are included to illustrate the use of the proposed method.

II. HAAR FUNCTIONS

The Haar functions form an orthogonal and orthonormal system of periodic square waves. The amplitude values of these square waves do not have uniform value, as with Walsh waveforms, but assume a limited set of values, 0, ± 1, $\pm\sqrt{2}$, ± 2, $\pm 2\sqrt{2}$ etc. If we consider the time base to be defined as $0 \le t \le 1$ then, the set of Haar functions is described as follows. And the waveform of first eight Haar functions is shown in Fig. 2-1.[4]

$$h(0,t) = 1 \quad \text{for} \quad 0 \le t \le 1$$

$$h(1,t) = \begin{cases} 1 & \text{for} \quad 0 \le t \le \frac{1}{2} \\ -1 & \text{for} \quad \frac{1}{2} \langle t \le 1 \end{cases}$$

$$h(2,t) = \begin{cases} \sqrt{2} & \text{for} \quad 0 \le t < \frac{1}{4} \\ -\sqrt{2} & \text{for} \quad \frac{1}{4} \le t < \frac{1}{2} \\ 0 & \text{for} \quad \frac{1}{2} \le t \le 1 \end{cases}$$

$$h(2^p+n,t) = \begin{cases} \sqrt{2^p} & \frac{n}{2^p} \le t < (n+\frac{1}{2})/2^p \\ -\sqrt{2^p} & (n+\frac{1}{2})/2^p \le t < (n+1)2^p \\ 0 & \text{elsewhere} \end{cases}$$

where p=1, 2..., n=0, 1.....2^p-1 (2-1)

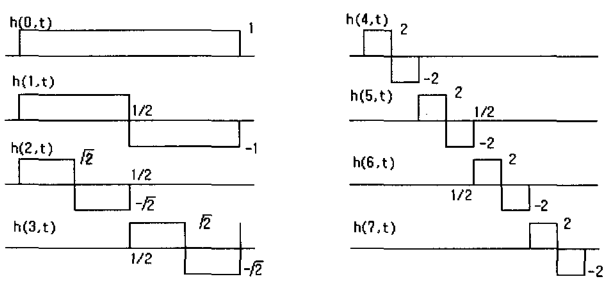

Fig. 2-1 The first eight Haar functions

Manuscript received July 22, 2004.

J.H. Park is with the Department of Information & Control Engineering, Chungju National University, Chungju-si, Chungbuk-do, Korea, 380-702. (email:jhpark@chungju.ac.kr)

T.S. Sidhu is with the Department of Electrical and Computer Engineering, Faculty of Engineering, University of Western Ontario, London, Ontario, Canada, N6A 5B9 (email:sidhu@eng.uwo.ca)

By adopting a different definition for the series, we can write Haar functions as follows:

$h(0,t)=1$ for $0 \le t \le 1$

$$h(i,j,t)=\begin{cases} \sqrt{2^i} & \text{for} \quad \frac{j-1}{2^i} \le t \le (j-\frac{1}{2})/2^i \\ -\sqrt{2^i} & \text{for} \quad (j-\frac{1}{2})/2^i < t \le \frac{j}{2^i} \\ 0 & \text{elsewhere} \end{cases} \quad (2\text{-}2)$$

Then the Haar functions can be referred to by order j and degree i as well as time t. The degree i denotes a subset having the same number of zero crossing in a given width $\frac{1}{2^i}$. The order j gives the position of the function within this subset.

From the definition in equation (2-1) and (2-2), it can be seen that the Haar functions are orthogonal, thus

$$\int_0^1 h(m,t)h(n,t) = \begin{cases} 1 & \text{for} \quad m=n \\ 0 & \text{for} \quad m \ne n \end{cases} \quad (2\text{-}3)$$

It is more convenient to express the waveform of Haar functions by matrix form. For instance, the first four Haar functions H_4 has the form

$$H_4 = \begin{bmatrix} H_0(t) \\ H_1(t) \\ H_2(t) \\ H_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \quad (2\text{-}4)$$

III. HAAR TRANSFORM

A. Haar transform

A function $f(t)$ is absolutely integrable on $t \in [0,1]$, then it can be expanded as an infinite series in term of Haar functions.[5]

$$f(t) = f_0 h_0(t) + f_1 h_1(t) + f_2 y_2(t) + \ldots\ldots = \sum_{i=0}^{\infty} f_i h_i(t) \quad (3\text{-}1)$$

Where f_i is the ith sequentially ordered coefficient of the Haar functions expansion of function $f(t)$ and h_i is the ith order Haar functions. The coefficient of the Haar functions expansion is given as equation (3-2)

$$f_i = \int_0^1 f(t) h_i(t) dt \quad (3\text{-}2)$$

Now an approximate transform of $f(t)$ in terms of the first n terms of the Haar functions and its matrix expression can be written as follows:

$$f(t) = \sum_{i=0}^{n-1} f_i h_i(t) = F_n^T H_n(t) \quad (3\text{-}3)$$

Where F_n is coefficient vector of $f(t)$ and $H_n(t)$ is its Haar functions vector. T denotes transposition.

B. Two dimension Haar functions approximation

To solve a partial differential equation of two variables two dimension Haar functions approximation concept is introduced newly. Consider a function $f(x,t)$ of two variables on $x \in [0,1]$ and $t \in [0,1]$. The Haar functions can approximately represent it with respect to t.

$$f(x,t) = \sum_{i=0}^{\infty} f_i(x) h_i(t) \quad (3\text{-}4)$$

We can approximate equation (3-4) as equation (3-5).

$$f(x,t) = \sum_{i=0}^{n-1} f_i(x) h_i(t) \quad (3\text{-}5)$$

Using the orthonormal property of Haar functions, the coefficient functions $f_i(t)$ of equation (3-5) is

$$f_i(x) = \int_0^1 f(x,t) h_i(t) dt \quad (3\text{-}6)$$

And Haar functions approximation of $f_i(t)$ gives

$$f_i(t) = \sum_{j=0}^{m-1} h_j(x) f_{ji} \quad (3\text{-}7)$$

where $h_j(x)$ are Haar functions with respect to x. The coefficients f_{ji} are obtained by

$$f_{ji} = \int_0^1 h_j(x) f_i(x) dx = \int_0^1 \int_0^1 f(x,t) h_i(t) h_j(x) dx dt \quad (3\text{-}8)$$

Therefore two variables function $f(x,t)$ can be approximated as equation (3-9) using the two dimension Haar functions approximation method.

$$f(x,t) = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} f_{ji} h_i(t) y_{j,i} h_j(x) = H_m^T(x) F_{mn} H_n(t) \quad (3\text{-}9)$$

C. operation matrix

The integrations of Haar functions with respect to time t form a ramp and triangular waveforms. And they are related approximately to the Haar functions matrix itself.[6],[7] We can describe integrals of Haar functions in mathematical form

$$\int_0^t H_n(t) dt \cong P_n H_n(t) \quad (3\text{-}10)$$

$$P_n = \begin{bmatrix} P_{(\frac{n}{2}\times\frac{n}{2})} & -\frac{1}{\sqrt{2}} n^{-2/3} H_{(\frac{n}{2}\times\frac{n}{2})} \\ \frac{1}{\sqrt{2}} n^{-2/3} H_{(\frac{n}{2}\times\frac{n}{2})} & 0_{(\frac{n}{2}\times\frac{n}{2})} \end{bmatrix} \quad (3\text{-}11)$$

where P_n is the operation matrix of Haar functions for integration. For example, P_2 is obtained as follows[8]:

$$P_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & 0 \end{bmatrix} \quad (3\text{-}12)$$

Repeated application of P_n for the m times repeated integration implies that

$$\int_0^t \int_0^t \int_0^t .....\int_0^t H_n(t)dt^m = P_n^m H_n(t) \quad (3\text{-}13)$$

IV. ESTIMATION OF NON-LINEAR DISTRIBUTED PARAMETER SYSTEMS

A. Linear DPS

Consider a simple linear DPS described by the following partial differential equation.

$$c_1 \frac{\partial y(x,t)}{\partial t} + c_2 \frac{\partial y(x,t)}{\partial x} + c_3 y(x,t) = u(x,t) \quad (4\text{-}1)$$

Where c_1, c_2 and c_3 are unknown system parameters to be identified. And $u(x,t)$ and $y(x,t)$ are two variables input and output functions.

We try to identify the system parameters of equation (4-1) by Haar transform for $x \in [0,1]$ and $t \in [0,1]$. Integration of equation (4-1) with respect to t and x gives

$$c_1 \int_0^x y(x,t)dx + c_2 \int_0^t y(x,t)dt + c_3 \int_0^t \int_0^x y(x,t)dxdt$$
$$- c_1 \int_0^x y(x,0)dx - c_2 \int_0^t y(0,t)dt = \int_0^t \int_0^x u(x,t)dxdt \quad (4\text{-}2)$$

where $y(x,0)$ and $y(0,t)$ are unknown initial and boundary condition.

According to the two dimension Haar functions approximation of equation (3-9), $y(x,t)$ and $u(x,t)$ are given by equation (4-3) and (4-4). U_{mn} is coefficient vector of $u(x,t)$ and Y_{mn} is coefficient vector of $y(x,t)$

$$u(x,t) = \sum_{j=0}^{m-1}\sum_{n=0}^{n-1} u_{ji} h_i(t) h_j(x) = H_m^T(x) U_{mn} H_n(t)$$
$$= H_n^T(t) U_{mn}^T H_m(x) \quad (4\text{-}3)$$

$$y(x,t) = \sum_{j=0}^{m-1}\sum_{i=0}^{n-1} y_{ji} h_i(t) h_j(x)$$
$$= H_m^T(x) Y_{mn} H_n(t) = H_n^T(t) Y_{mn}^T H_m(x) \quad (4\text{-}4)$$

Also, we can approximate the unknown $y(x,0)$ and $y(0,t)$ using Haar transform as follows:

$$y(x,0) = \sum_{i=0}^{q-1} d_i h_i(x) = \sum_{i=0}^{q-1} d_i H_m^T(x) F_{i+1,1} H_n(t) \quad (4\text{-}5)$$

$$y(0,t) = \sum_{i=0}^{p-1} e_i h_i(t) = \sum_{i=0}^{p-1} e_i H_m^T(x) F_{1,i+1} H_n(t) \quad (4\text{-}6)$$

Where F_{ij} is an matrix having the (i,j)th element unity and the remaining elements zero. d_i is coefficient vector of $y(x,0)$ and e_i is coefficient vector of $y(0,t)$.

Substituting (4-3)-(4-6) into (4-2), then equation (4-2) can be arranged and written as following form:

$$H_m^T(x)[c_1 P_m^T Y_{mn} + c_2 Y_{mn} P_n + c_3 P_m^T Y_{mn} P_n$$
$$- c_1 \sum_{i=0}^{q-1} d_i P_m^T F_{i+1,1} - c_2 \sum_{i=0}^{p-1} e_i F_{1,i+1} P_n] H_n(t)$$
$$= H_m^T(x)[P_m^T U_{mn} P_n] H_n(t) \quad (4\text{-}7)$$

Applying the orthonormal property of Haar functions, we can simplify equation (4-7).

$$c_1 P_m^T Y_{mn} + c_2 Y_{mn} P_n + c_3 P_m^T Y_{mn} P_n$$
$$- c_1 \sum_{i=0}^{q-1} d_i P_m^T F_{i+1,1} - c_2 \sum_{i=0}^{p-1} e_i F_{1,i+1} P_n = P_m^T U_{mn} P_n \quad (4\text{-}8)$$

Then equation (4-8) can be written in the form

$$K\theta = r \quad (4\text{-}9)$$

where K is the $mn \times (c_1 + c_2 + c_3 + q + p)$ matrix of known elements, θ is the vector of unknowns and r is the mn vector of known elements.

$$K = \begin{bmatrix} (P_m^T Y_{mn})_1 & (Y_{mn} P_n)_1 & (P_m Y_{mn} P_n)_1 & (F_{1,1} P_n)_1 & . & (F_{q,1} P_n)_1 & (F_{1,1} P_n)_1 & . & (F_{1,p} P_n)_1 \\ (P_m^T Y_{mn})_2 & (Y_{mn} P_n)_2 & (P_m Y_{mn} P_n)_2 & (F_{1,1} P_n)_2 & . & (F_{q,1} P_n)_2 & (F_{1,1} P_n)_2 & . & (F_{1,p} P_n)_2 \\ : & : & : & : & . & : & : & . & : \\ (P_m^T Y_{mn})_{n-1} & (Y_{mn} P_n)_{n-1} & (P_m Y_{mn} P_n)_{n-1} & (F_{1,1} P_n)_{n-1} & . & (F_{q,1} P_n)_{n-1} & (F_{1,1} P_n)_{n-1} & . & (F_{1,p} P_n)_{n-1} \\ (P_m^T Y_{mn})_n & (Y_{mn} P_n)_{n1} & (P_m Y_{mn} P_n)_n & (F_{1,1} P_n)_n & . & (F_{q,1} P_n)_n & (F_{1,1} P_n)_n & . & (F_{1,p} P_n)_n \end{bmatrix} \quad (4\text{-}10)$$

$$\theta^T = [c_1 \quad c_2 \quad c_3 \quad -c_1 \sum_{i=0}^{q-1} d_i \quad -c_2 \sum_{i=0}^{p-1} e_i]'$$

$$r = \begin{bmatrix} (P_m^T U_{mn} P_n)_1 \\ (P_m^T U_{mn} P_n)_2 \\ : \\ (P_m^T U_{mn} P_n)_{n-1} \\ (P_m^T U_{mn} P_n)_n \end{bmatrix} \quad (4\text{-}11)$$

Where $(\bullet)_i$ denotes an ith vector of the matrix K and r. The equation (4-9) can be solved with the least square method as following equation (4-12). In this case, $K^T K$ is invertible.

$$\hat{\theta} = (K^T K)^{-1} K^T r \cdot \quad (4\text{-}12)$$

The solution directly gives the system parameters c_1, c_2 and c_3, while the Haar functions coefficients are determined. And then we can estimate the initial and boundary condition in equation (4-5) and (4-6) using the determined d_i and e_i.

B. Non-linear DPS

For a non-linear DPS estimation case, consider the following partial differential equation model.

$$c_5 \frac{\partial^2 y^{p5}(x,t)}{\partial t^2} + c_4 \frac{\partial^2 y^{p4}(x,t)}{\partial x \partial t} + c_3 \frac{\partial^2 y^{p3}(x,t)}{\partial x^2} + c_2 \frac{\partial y^{p2}(x,t)}{\partial t} + c_1 \frac{\partial y^{p1}(x,t)}{\partial x} + c_0 y^{p0}(x,t) = u^{p6}(x,t) \quad (4\text{-}13)$$

To identify equation (4-13), the previous approach algorithm will be applied to the equation. We can integrate twice the equation (4-13) with respect to x and t, and then we can get equation (4-14).

$$\begin{aligned} & c_5 \int_0^x \int_0^x y^{p5}(x,t)dxdx + c_4 \int_0^x \int_0^t y^{p4}(x,t)dtdx \\ & + c_3 \int_0^t \int_0^t y^{p3}(x,t)dtdt + c_2 \int_0^x \int_0^x \int_0^t y^{p2}(x,t)dtdxdx \\ & + c_1 \int_0^x \int_0^x \int_0^t y^{p1}(x,t)dtdxdx \\ & + c_0 \int_0^x \int_0^x \int_0^t \int_0^t y^{p0}(x,t)dtdtdxdx + \int_0^x \int_0^t \int_0^t \alpha(x)dtdxdx \\ & + \int_0^x \int_0^t \int_0^t \beta(x)dtdxdx + \int_0^x \int_0^x \int_0^t \gamma(x)dtdxdx \\ & - c_5 \int_0^x \int_0^x y^{p5}(x,0)dxdx - c_3 \int_0^t \int_0^t y^{p3}(0,t)dtdt \\ & = \int_0^x \int_0^x \int_0^t \int_0^t u^{p6}(x,t)dtdtdxdx \end{aligned} \quad (4\text{-}14)$$

where

$$\alpha(x) = -c_5 \left[\frac{\partial y^{p5}(x,t)}{\partial t} \right]_{t=0} - c_2 y^{p2}(x,0) \quad (4\text{-}15)$$

$$\beta(x) = -c_4 \left[\frac{\partial y^{p4}(x,t)}{\partial x} \right]_{t=0} \quad (4\text{-}16)$$

$$\gamma(x) = -c_3 \left[\frac{\partial y^{p3}(x,t)}{\partial x} \right]_{x=0} - c_1 y^{p1}(0,t) \quad (4\text{-}17)$$

Determining the system parameters c_0, c_1, c_2, c_3, c_4, c_5 and the initial and boundary conditions is the estimation problem for a non-linear distributed parameter system. Now expanding the different functions into the Haar transform and the two dimension Haar functions approximation method yields,[9]

$$y^{pz}(x,t) = H_m^T(x) Q_z H_n(t) \quad (4\text{-}18)$$

$$u^{pz}(x,t) = H_m^T(x) R_z H_n(t) \quad (4\text{-}19)$$

where Q_z is a mn matrix given by equation (4-20)

$$Q_z = z\text{th column of } Q = \sum_{i=1}^{m} \eta_{i-1}^{(m)} Y_{mn} \eta_{z-1}^{(n)} Y_i^T \quad (4\text{-}20)$$

where $\eta_i^{(m)} = \begin{bmatrix} \eta_i^{(m/2)} & 0^{(m/2)} \\ 0^{(m/2)} & \eta_i^{(m/2)} \end{bmatrix}$ and Y_i^T is ith column of matrix Y_{mn}.

$$y^{p5}(x,0) = \sum_{i=0}^{q-1} g_i h_i(x) = \sum_{i=0}^{q-1} g_i H_m^T(x) F_{i+1,1} H_n(t) \quad (4\text{-}21)$$

$$y^{p3}(0,t) = \sum_{i=0}^{p-1} l_i h_i(t) = \sum_{i=0}^{p-1} l_i H_m^T(x) F_{1,i+1} H_n(t) \quad (4\text{-}22)$$

$$\alpha(x) = \sum_{i=0}^{k-1} \alpha_i H_m^T(x) F_{i+1,1} H_n(t), \quad (k \le m) \quad (4\text{-}23)$$

$$\beta(x) = \sum_{i=0}^{l-1} \beta_i H_m^T(x) F_{i+1,1} H_n(t), \quad (l \le m) \quad (4\text{-}24)$$

$$\gamma(x) = \sum_{i=0}^{r-1} \gamma_i H_m^T(x) F_{1,i+1} H_n(t), \quad (r \le n) \quad (4\text{-}25)$$

Where F_{ij} is an matrix having the (i,j)th element unity and the remaining elements zero. Equation (4-18)-(4-25) are substituted in equation (4-14). Thus we can get the equation (4-26) as Haar transformed form.

$$\begin{aligned} & c_5 (P_m^T)^2 Q_5 + c_4 P_m^T Q_4 P_n + c_3 Q_3 P_n^2 + c_2 (P_m^T)^2 Q_2 P_n \\ & + c_1 P_m^T Q_1 P_n^2 + c_0 (P_m^T)^2 Q_0 P_n^2 + \sum_{i=0}^{k-1} \alpha_i (P_m^T)^2 F_{i+1,1} P_n \\ & + \sum_{i=0}^{l-1} \beta_i P_m^T F_{i+1,1} P_n^2 \sum_{i=0}^{\gamma-1} P_m^T)^2 F_{1,i+1} P + \sum_{i=0}^{\sigma-1} g^{\sim}_i (P_m^T)^2 F_{i+1,1} \\ & + \sum_{i=0}^{v-1} l_i^{\sim} F_{1,i+1} P_n^2 = (P_m^T)^2 R_6 P_n^2 \end{aligned} \quad (4\text{-}26)$$

where $g^{\sim}_i = -c_5 g_i, l^{\sim}_i = -c_3 l_i$

Equation (4-26) can be written in the similar form of equation (4-9).

$$K\theta = r \quad (4\text{-}27)$$

Now, we can solve equation (4-27) as the same manner of linear DPS in (4-12). Therefore the proposed method yields an estimate for the both the coefficients of the system and its initial and boundary conditions of a non-linear DPS. Numerical examples are shown for supporting the method. Clearly the present approach algorithm can be applied to the case of higher-order partial differential equation models.

V. NUMERICAL EXAMPLES

A. Example (1):

Consider the following linear DPS which is described by the first-order partial differential equation.[10],[11]

$$c_1 \frac{\partial y(x,t)}{\partial t} + c_2 \frac{\partial y(x,t)}{\partial x} + c_3 y(x,t) = u(x,t) \quad (5\text{-}1)$$

$$u(x,t) = xt + 4x + 2t, \quad y(x,t) = xt \quad (5\text{-}2)$$

$$y(x,0) = y(0,t) = 0 \tag{5-3}$$

From the exact solutions of equation (5-2) and (5-3), we can get the output mean value Y_{dji} for $y(x,t)$ and input mean value U_{dji} for $u(x,t)$ using rectangular geometry concept with respect to x and t. Where $i=j=0, 1, .., m-1$ and $m(n)$ denotes term of Haar transform. Choosing $m=n=4$, $\Delta x = \Delta t = 0.25$ and $i=j=0$, 1, 2, 3. Thus Y_{dji} and U_{dji} are given by following table 5.1

Table 5.1. input and output mean values of example (1)

$\frac{3}{4} \le x \le 1$	$\frac{0.0068}{0.1475}$	$\frac{0.0205}{0.2236}$	$\frac{0.0342}{0.2998}$	$\frac{0.0478}{0.3760}$
$\frac{1}{2} \le t < \frac{3}{4}$	$\frac{0.0049}{0.1143}$	$\frac{0.0146}{0.1885}$	$\frac{0.0244}{0.2588}$	$\frac{0.0342}{0.3311}$
$\frac{1}{4} \le t < \frac{1}{2}$	$\frac{0.0029}{0.0772}$	$\frac{0.0088}{0.1494}$	$\frac{0.0146}{0.2178}$	$\frac{0.0205}{0.2881}$
$0 \le t < \frac{1}{4}$	$\frac{0.0010}{0.0439}$	$\frac{0.0029}{0.1123}$	$\frac{0.0049}{0.1768}$	$\frac{0.0068}{0.2432}$
Y_{dij} / U_{dij}	$0 \le x < \frac{1}{4}$	$\frac{1}{4} \le x < \frac{1}{2}$	$\frac{1}{2} \le x < \frac{3}{4}$	$\frac{3}{4} \le x \le 1$

Now, to identify the system parameters, c_1, c_2 and c_3, we can apply the proposed identification method. Convert the equation (5-1) to an integral one as follows:

$$c_1 \int_0^x y(x,t)dx + c_2 \int_0^t y(x,t)dt + c_3 \int_0^t \int_0^x y(x,t)dxdt = \int_0^t \int_0^x u(x,t)dxdt \tag{5-4}$$

And then, expand the equation (5-4) by double Haar series approximation and its operation matrix.

$$c_1 P_m^T Y_{mn} + c_2 Y_{mn} P_n + c_3 P_m^T Y_{mn} P_n = P_m^T U_{mn} P_n \tag{5-5}$$

We can obtain the coefficient of Haar transform Y_{mn} and U_{mn} from the given Y_{dji} and U_{dji} respectively.

$$U_{mn} = \frac{1}{m}(H_m^T)^{-1} U_{dij} H^T \tag{5-6}$$

$$Y_{mn} = \frac{1}{m}(H_m^T)^{-1} Y_{dij} H^T \tag{5-7}$$

Also, we know H_4 and P_4 as follows:

$$H_m = H_n = H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$$P_4 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{\sqrt{2}}{16} & -\frac{\sqrt{2}}{16} \\ \frac{1}{4} & 0 & -\frac{\sqrt{2}}{16} & -\frac{\sqrt{2}}{16} \\ \frac{\sqrt{2}}{16} & \frac{\sqrt{2}}{16} & 0 & 0 \\ \frac{\sqrt{2}}{16} & \frac{\sqrt{2}}{16} & 0 & 0 \end{bmatrix} \tag{5-8}$$

According to equation (4-9) and (4-10), we can decide matrix K, r and θ in equation (4-8) for solving the problem.
Where K is the $4 \times 4 \times (c_1 + c_2 + c_3)$ matrix of known elements, θ is the vector of unknowns and r is the 4×4 vector of known elements.

$$K = \begin{bmatrix} (P_2^T Y_{22})_1 & (Y_{22} P_2)_1 & (P_2^T Y_{22} P_2)_1 \\ \vdots & \vdots & \vdots \\ (P_2^T Y_{22})_{16} & (Y_{22} P_2)_{16} & (P_2^T Y_{22} P_2)_{16} \end{bmatrix} \tag{5-9}$$

$$\theta^T = [c_1 \quad c_2 \quad c_3], \quad r = \begin{bmatrix} (P_2^T U_{22} P_2)_1 \\ \vdots \\ (P_2^T U_{22} P_2)_{16} \end{bmatrix} \tag{5-10}$$

The results of K and r are given as equation (5-11)

$$K = \begin{bmatrix} 0.0054 & 0.0054 & 0.0018 \\ -0.0039 & -0.0027 & -0.0013 \\ \vdots & \vdots & \vdots \\ 0.0004 & 0.0004 & 0.0003 \end{bmatrix} \quad r = \begin{bmatrix} 0.0338 \\ -0.0198 \\ \vdots \\ 0.0025 \end{bmatrix} \tag{5-11}$$

Apply equation (4-12) to get square matrix and decide $\hat{\theta}$ for [c_1 c_2 c_3]. Therefore we can identify the first-order DPS's parameters c_1, c_2 and c_3 finally.

$$\hat{\theta} = (K^T K)^{-1} K^T r = \begin{bmatrix} 1.9585 \\ 3.9155 \\ 1.2270 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{5-12}$$

As shown in equation (5-12), c_1=1.9585, c_2 =3.9155 and c_3=1.2270 are obtained. The results agree precisely with the exact values of c_1=2, c_2=4 and c_3=1. If the term of transform $m(n)$ is increased, the results will be more close to the exact values.

B. Example (2):

Following non-linear DPS is considered.[10] Given a record of $y(x,t)$and $u(x,t)$, the problem is to estimate the parameters of the system. For this purpose, c_1=2, c_2=2, c_3=1 and m=n=4 are taken.

$$c_1 \frac{\partial y(x,t)}{\partial x} + c_2 \frac{\partial y^2(x,t)}{\partial t} + c_3 y(x,t) = u(x,t) \tag{5-13}$$

$$y(x,0)=y(0,t)=0, \quad y(1,t)=t \tag{5-14}$$

And the input output functions are as follows:

$$u(x,t) = 4x^2 t + 2t + xt, \quad y(x,t) = xt \tag{5-15}$$

To apply the proposed method, we integrate equation (5-13) with respect to x and t and expand the integrating results using Haar transform. Then we can convert the partial differential equation to the following algebraic equation (5-15) simply

$$c_1\int_0^x y(x,t)dx + c_2\int_0^x y^2(x,t)dx + c_3\int_0^t\int_0^x y(x,t)dxdt = \int_0^t\int_0^x u(x,t)dxdt \quad (5\text{-}16)$$

And we can write the equation using the presented method as follows:

$$c_1 P_m^T Y_{mn} + c_2 Q_2 P_n + c_3 P_m^T Y_{mn} P_n = P_m^T U_{mn} P_n \quad (5\text{-}17)$$

Now, we can solve equation (5-17) as the same manner of linear DPS in example (1). Therefore the proposed method yields an estimate for the both the coefficient of the system and its initial conditions of a non-linear DPS. Using the procedure outlined above, we get the estimation values of the parameters c_1=2.0866, c_2=2.0394, c_3=0.9653. The proposed method is very simple and accurate.

VI. CONCLUSIONS

Estimating the parameters, initial and boundary conditions of non-linear distributed parameter system using Haar functions and its transform has been presented in this paper. The two dimension Haar functions approximation method is introduced and applied newly to solve a partial differential equation and a non-linear DPS problem. Applying the proposed method, a partial differential equation can be converted into an algebraic equation, and thus the estimation of non-linear distributed parameter system procedure is either greatly reduced or much simplified accordingly.

REFENENCES

[1] P.N. Paraskevopoulos and A.C. Bounas, 'Distributed parameter system identification via Walsh functions', Int. J. Systems Sci, vol. 1, no., pp. 75-83, 1978.

[2] S.G. Tzafestas and P. Stavroulakis, 'Walsh and block pulse operational matrices in distributed parameter and delay system', Digital techniques in simulation, communication and control, Elsevier Science Publishers, North-Holland, 1985.

[3] M.S.P. Sinha, V.S. Rajamani and A.K. Sinha 'Identification of non-linear distributed systems using Walsh functions', Int. J. Control, vol. 32, no. 4, pp. 669-676, 1980.

[4] Beauchamp, 'Walsh functions and their applications', Academic Press, London, 1975.

[5] John E. Shore, 'On the application of Haar functions', IEEE Trans. on Commun., pp. 209-216, 1973.

[6] Masaaki Ohkita and Yasuhiro Kobayashi, 'An application of rationalized Haar functions to solution of linear differential equations', IEEE Trans. on Circuit and Systems, vol. CAS-33, no. 9, pp. 853-862, 1986.

[7] C.F. Chen and C.H. Hsiao, 'Haar wavelet method for solving lumped and distributed parameter systems', IEE Proc. Control Theory Appl., vol. 144, no.1, pp. 87-94, 1997.

[8] D.S. Ahn, 'Application of orthogonal functions', Bokdu Press, Seoul, 1997.

[9] N.S. Hsu and B. Cheng, 'Identification of non-linear distributed systems via block pulse functions', Int. J. Control, vol. 36, no. 2, pp. 281-291, 1982.

[10] J. Cooper, 'Introduction to partial differential equations with Matlab', Brichauser, Boston, 1998.

[11] S. Omatu and J.H. Senfeld, 'Distributed parameter systems', Clarendon Press, Oxford, 1989.

Joon-Hoon Park
received the B.S. degree in electrical engineering from SungKyunKwan University in 1981 and the M.S. and Ph.D. degree in control engineering from SKK Univ., in 1983 and 1992, respectively. He was a senior researcher of the R&D Center, Samsung Electronics Co. from 1983 to 1990. Since 1991, he is currently Professor of the Dept. of Information and Control Engineering, Chungju National University. Now he is a Post-Doc. Fellow of the University of Western Ontario, Canada. He is interested in optimal control theory, system design and its applications.

T. S. Sidhu
Received the B.E. degree from the Punjabi University, India, in 1979 and the M.Sc. and Ph.D. degree from the Univ. of Saskatchewan, Saskatoon, SK, Canada, in 1985 and 1989, respectively. He is currently Professor and the Hydro One Chair in Power System Engineering at the University of Western Ontario Canada. His areas of research interests are power system protection, monitoring and control.