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# THE E-EULER PROCESS FOR NONAUTONOMOUS SYSTEMS

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ABSTRACT. The E-Euler process has been proposed for autonomous dynamical systems in [7]. In this paper, the E-Euler process is extended to nonautonomous dynamical systems. When a discrete function is bounded or gradually decreases to $\epsilon << 1$ as $n \to \infty$, it is shown that the relative error converges to a constant or decreases.

## 1. INTRODUCTION

The nonlinear autonomous systems of the form

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t)) \equiv J\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^m, \quad t \geq 0, \tag{1.1}$$

where $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $J = \frac{\partial}{\partial \mathbf{x}}\mathbf{f}(\mathbf{0})$, are considered in [7] and [8]. The eigenvector matrix of $J$, the s-matrix, the s-transformed system, and the exponential Euler process have been introduced in [7] and [8] as follows :

(1) The eigenvector matrix $P$ is given by

$$P = [\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}_1, \mathbf{v}_{p+1}, \ldots, \mathbf{w}_q, \mathbf{v}_{p+q}],$$

where $\mathbf{v}_j\,(j = 1, \ldots, p)$ and $\mathbf{v}_{p+k} \pm i\mathbf{w}_k\,(k = 1, \ldots, q)$ are eigenvectors of $J$.

(2) The s-matrix is given by

$$S \equiv \alpha I + \frac{1}{2}(\widehat{S} - \widehat{S}^T), \tag{1.2}$$

where $P^{-1}JP = \widehat{S} + \widehat{N}$, $\widehat{S}$ is semisimple, $\widehat{N}$ is nilpotent, $\widehat{S}\widehat{N} = \widehat{N}\widehat{S}$ and $\alpha = \alpha[J]$ is the largest real part of eigenvalues of $J$.

(3) By $\mathbf{x}(t) = P\mathbf{y}(t)$, the problem (1.1) is transformed into

$$\mathbf{y}'(t) = S\mathbf{y}(t) + \mathbf{u}(\mathbf{y}(t)), \quad \mathbf{y}(0) = P^{-1}\mathbf{x}_0, \tag{1.3}$$

where $\mathbf{u}(\mathbf{y}) = (\widehat{S} - S + \widehat{N})\mathbf{y} + P^{-1}\mathbf{g}(P\mathbf{y})$.

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For $S \neq O$, the system (1.3) or (1.1) is called an *exponentially dominant system*, if the solution to the nonlinear system (1.3) asymptotically follows the solution to the corresponding linear part $\mathbf{y}'(t) = S\mathbf{y}(t)$ as $t$ tends to infinity.

(4) Applying $\mathbf{y}(t) = \exp(tS)\mathbf{z}(t)$ to the exponentially dominant system (1.3), we have an initial value problem for $\mathbf{z}(t)$ ([3]):

$$\mathbf{z}'(t) = \exp(-tS)\mathbf{u}(\exp(tS)\mathbf{z}(t)), \quad \mathbf{z}(0) = P^{-1}\mathbf{x}_0. \tag{1.4}$$

(5) By the Euler method and $\mathbf{z}_n = \exp(-t_nS)\mathbf{y}_n$, the *E-Euler process* (exponential Euler process) is obtained:

$$\mathbf{y}_{n+1} = \exp(hS)\Big\{\mathbf{y}_n + h\mathbf{u}(\mathbf{y}_n)\Big\}, \quad \mathbf{x}_{n+1} = P\mathbf{y}_{n+1}, \quad t_{n+1} = t_n + h. \tag{1.5}$$

It is shown that

(1) The E-Euler process is efficient to the exponentially dominant systems (see [7] and [8]).
(2) Since the matrix exponential $\exp(hS)$ is exactly computed (see [5],[6]), the process (1.5) is based on the precise computation of matrix exponential.
(3) In [7], the implementation of E-Euler process has been discussed and the process is compared with RKSUITE which is developed by Brankin, Gladwell and Shampine [1].
(4) The E-Euler process nicely recover the long term behavior for oscillatory problems than the classical Runge-Kutta method and RKSUITE.
(5) In [8], the E-Euler process is unconditionally contractive on some subclasses of the dissipative exponentially dominant systems, and unconditionally noncontractive on some subclasses of the nondissipative exponentially dominant systems.
(6) In [8], the relative errors of the approximation obtained by the E-Euler process can be estimated by a function $\exp(-tS)\mathbf{u}(\mathbf{y}(t))$.

In this paper, we will extend the E-Euler process (1.5) to the nonautonomous dynamical systems.

## 2. Extension of the process

Let us consider the nonautonomous dynamical systems

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2.1}$$

where $\mathbf{f}(t,\cdot) : M_t \to \mathbb{R}^m$ is twice continuously differentiable with respect to $\mathbf{x}$, and $M_t \subset \mathbb{R}^m$ is the convex region which is defined in [2]. We assume that $\frac{\partial}{\partial \mathbf{x}}\mathbf{f}(0,\mathbf{0})$ does not vanish.

Since $\mathbf{f}(t,\mathbf{x}(t))$ is twice continuously differentiable with respect to $\mathbf{x}$, it is represented as

$$\mathbf{f}(t,\mathbf{x}(t)) = \mathbf{f}(t,\mathbf{0}) + \frac{\partial}{\partial \mathbf{x}}\mathbf{f}(t,\mathbf{0})\mathbf{x}(t) + \mathbf{r}(t,\mathbf{x}(t)),$$

where $\mathbf{r}(t, \mathbf{x}(t))$ is the remainder. Then (2.1) is divided into two parts:

$$\mathbf{x}'(t) = \tilde{J}\mathbf{x}(t) + \mathbf{g}(t, \mathbf{x}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2.2}$$

where the $(i, j)$ component of $\tilde{J}$ is the constant term of the $(i, j)$ component of $\frac{\partial}{\partial \mathbf{x}}\mathbf{f}(t, \mathbf{0})$ and

$$\mathbf{g}(t, \mathbf{x}(t)) = \mathbf{f}(t, \mathbf{0}) + \Big(\frac{\partial}{\partial \mathbf{x}}\mathbf{f}(t, \mathbf{0}) - \tilde{J}\Big)\mathbf{x}(t) + \mathbf{r}(t, \mathbf{x}(t)).$$

For an eigenvector matrix $P$ of $\tilde{J}$, the real canonical form of $\tilde{J}$ is given by

$$P^{-1}\tilde{J}P = \bar{S} + \bar{N}, \tag{2.3}$$

where $\bar{S}$ is a canonical semisimple matrix, $\bar{N}$ is nilpotent and $\bar{S}\bar{N} = \bar{N}\bar{S}$. Then, an s-matrix $\tilde{S}$ is given by

$$\tilde{S} = \alpha I + \frac{1}{2}\Big(\bar{S} - \bar{S}^T\Big), \quad \text{where} \quad \alpha = \alpha[\tilde{J}].$$

By using $\mathbf{x}(t) = P\mathbf{y}(t)$ and $\tilde{S}$, the problem (2.2) is transformed to

$$\mathbf{y}'(t) = \tilde{S}\mathbf{y}(t) + \mathbf{u}(t, \mathbf{y}(t)), \quad \mathbf{y}(0) = P^{-1}\mathbf{x}_0, \tag{2.4}$$

where

$$\mathbf{u}(t, \mathbf{y}) = \Big(P^{-1}\tilde{J}P - \tilde{S}\Big)\,\mathbf{y}(t) + P^{-1}\,\mathbf{g}(t, P\mathbf{y}(t)).$$

The system (2.4) is called an *s-transformed system* of (2.1). For $\tilde{S} \neq O$, the system (2.1) or (2.4) is called an *exponentially dominant system*, if the solution to the nonlinear system (2.4) asymptotically follows the solution to the corresponding linear part $\mathbf{y}'(t) = \tilde{S}\mathbf{y}(t)$ as $t$ tends to infinity.

Applying $\mathbf{y}(t) = \exp(t\tilde{S})\mathbf{z}(t)$ to the exponentially dominant system (2.4), we have an initial value problem for $\mathbf{z}(t)$

$$\mathbf{z}'(t) = \exp(-t\tilde{S})\mathbf{u}(t, \exp(t\tilde{S})\mathbf{z}(t)), \quad \mathbf{z}(t_n) = \exp(-t_n\tilde{S})\mathbf{y}_n. \tag{2.5}$$

Apply Euler method to (2.5). Then we have

$$\mathbf{z}_{n+1} = \mathbf{z}_n + h\exp(-t_n\tilde{S})\mathbf{u}(t_n, \exp(t_n\tilde{S})\mathbf{z}_n). \tag{2.6}$$

Applying $\mathbf{z}_n = \exp(-t_n\tilde{S})\mathbf{y}_n$ to (2.6), we arrive at

$$\mathbf{y}_{n+1} := \exp(h\tilde{S})\,\big\{\mathbf{y}_n + h\mathbf{u}(t_n, \mathbf{y}_n)\big\}. \tag{2.7}$$

Finally, the numerical solution of (2.1) is obtained by

$$\mathbf{x}_n = P\mathbf{y}_n \qquad \text{for} \quad n = 1, 2, 3, \cdots.$$

Such a process is called an exponential Euler process (E-Euler process) for nonautonomous systems.

## 3. Numerical examples

In order to test the accuracy of the exponential Euler process, we consider the *relative error* (see [8]) defined by the elliptic vector norm in $\mathbf{x}$-space and the Euclidean norm in $\mathbf{y}$-space:

$$\rho_n = \rho(n;h) \equiv \frac{|\mathbf{x}(t_n) - \mathbf{x}_n|_P}{|\mathbf{x}(t_n)|_P} = \frac{|\mathbf{y}(t_n) - \mathbf{y}_n|}{|\mathbf{y}(t_n)|}, \tag{3.1}$$

and the linear part of $\mathbf{u}(t, \mathbf{y}(t))$:

$$\mathbf{w}(t, \mathbf{y}(t)) = \Big(P^{-1}\frac{\partial}{\partial \mathbf{x}}\mathbf{f}(t, \mathbf{0})P - \tilde{S}\Big)\mathbf{y}(t). \tag{3.2}$$

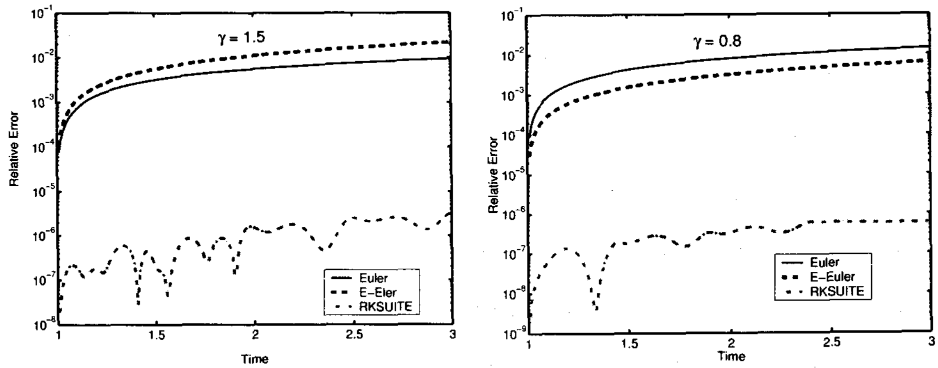
Two examples are tested by the Euler method (Euler), the E-Euler process (E-Euler) and Template 3a of RKSUITE. The programs were compiled using Visual Fortran and executed on a personal computer (Pentium III) with smallest positive number approximately 2.223D-308 and the unit of roundoff approximately 1.0D-16.

3.1. **Problem 1** ([4]). Consider the nonautonomous linear system

$$\mathbf{x}' = A(t)\mathbf{x} = \begin{pmatrix} -1 + \gamma\cos^2(t) & 1 - \gamma\sin(t)\cos(t) \\ -1 - \gamma\sin(t)\cos(t) & -1 + \gamma\sin^2(t) \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(t_0) = \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}. \tag{3.3}$$

The eigenvalue of $A(t)$ are time independent and given by

$$\lambda_{\pm} = \frac{1}{2}(\gamma - 2 \pm \sqrt{\gamma^2 - 4}).$$

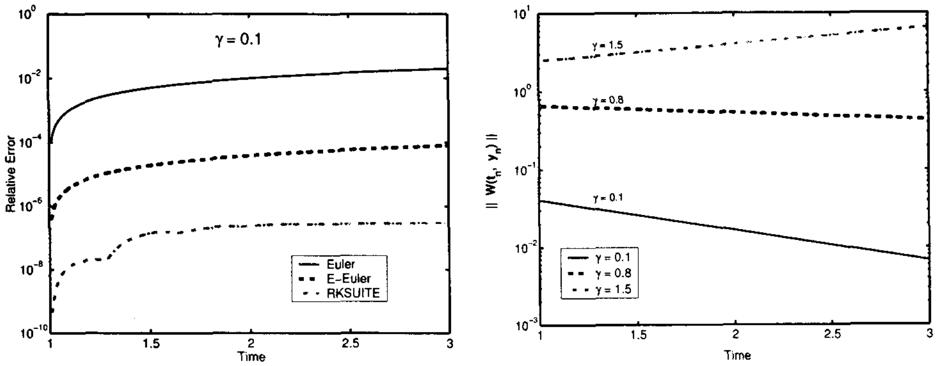



(a) $\gamma = 1.5$ (b) $\gamma = 0.8$

FIGURE 1. Graphs of $\rho(n; 0.01)$.

We have $Re(\lambda_{\pm}) < 0$ if $\gamma < 2$. Next, we apply a coordinate transformation such that the new coordinate frame rotate as time $t$ evolves:

$$\mathbf{y}(t) = Q(t)\mathbf{x}(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\mathbf{y}.$$




(a) $\gamma = 0.1$ (b) $\mathbf{w}(t_n, \mathbf{y}_n)$

FIGURE 2. Graphs of $\rho(n; 0.01)$ and $\mathbf{w}(n; 0.01)$.

Then, $\mathbf{y}(t)$ satisfies the equation

$$\mathbf{y}' = Q(t)A(t)Q^{-1}(t)\mathbf{y} = \begin{pmatrix} \gamma - 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y}. \tag{3.4}$$

So, the solution of (3.3) is given by

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{(\gamma-1)t} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}.$$

Hence, $\mathbf{y}$, and thus $\mathbf{x}$, is Lyapunov stable for $\gamma = 1$. For $\gamma < 1$ we even find asymptotic stability, while for $\gamma > 1$ the system is unstable. The real part of $\lambda_\pm$, given above, did not suggest this.

For the system (3.3), take the s-matrix $\tilde{S}$ and its eigenvector matrix $P$ as follows:

$$\tilde{S} = \tilde{J} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad P = I. \tag{3.5}$$

Then, the s-transformed system of (3.3) is given by

$$\mathbf{y}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} \gamma \cos^2(t) & -\gamma \sin(t)\cos(t) \\ -\gamma \sin(t)\cos(t) & \gamma \sin^2(t) \end{pmatrix} \mathbf{y}, \tag{3.6}$$

and the discrete function of the linear part of $\mathbf{u}(t, \mathbf{y}(t))$ is given by

$$\mathbf{w}(n; h) \equiv \begin{pmatrix} \gamma \cos^2(t_n) & -\gamma \sin(t_n)\cos(t_n) \\ -\gamma \sin(t_n)\cos(t_n) & \gamma \sin^2(t_n) \end{pmatrix} \mathbf{y}_n.$$

The problem is examined for $h = 0.01$, $\alpha = -1.0$, $\beta = -1.0$, $c_1 = 1.0$, $c_2 = 1.0$, $t_0 = 1.0$, $x_{1,0} = x_1(t_0)$ and $x_{2,0} = x_2(t_0)$. If the discrete function $\mathbf{w}(n; 0.01)$ is gradually increases (see Figure 2 (b) for $\gamma = 1.5$), the relative error of E-Euler is greater than Euler (see Figure 1 (a)). If the discrete function is gradually decreases (see Figure 2 (b) for $\gamma = 0.8$ and $\gamma = 0.1$), the relative error of E-Euler is smaller than Euler (see Figure 1 (b) and Figure 2 (a)).

3.2. **Problem 2.** Consider a nonlinear nonautonomous system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{t}\cos^2 t + \frac{1}{t}\sin^2 t & \frac{1}{t}\sin t\cos t - 1 \\ \frac{1}{t}\sin t \cos t + 1 & \frac{2}{t}\sin^2 t + \frac{1}{t}\cos^2 t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{t^2}\cos t(x_1\cos t + x_2 \sin t)^2 - \sin t \\ \frac{1}{t^2}\sin t(x_1 \cos t + x_2 \sin t)^2 + \cos t \end{pmatrix}.$$

The exact solution of the above equation is given by

$$x_1(t) = \left(\frac{t^2}{a-t}\right)\cos t - \left(t\left\{\ln t + b\right\}\right)\sin t,$$
$$x_2(t) = \left(\frac{t^2}{a-t}\right)\sin t + \left(t\left\{\ln t + b\right\}\right)\cos t.$$

Since $\frac{\partial}{\partial \mathbf{x}}\mathbf{f}(t,\mathbf{0}) = \begin{pmatrix} \frac{2}{t}\cos^2 t + \frac{1}{t}\sin^2 t & \frac{1}{t}\sin t\cos t - 1 \\ \frac{1}{t}\sin t \cos t + 1 & \frac{2}{t}\sin^2 t + \frac{1}{t}\cos^2 t \end{pmatrix}$, we have $\tilde{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

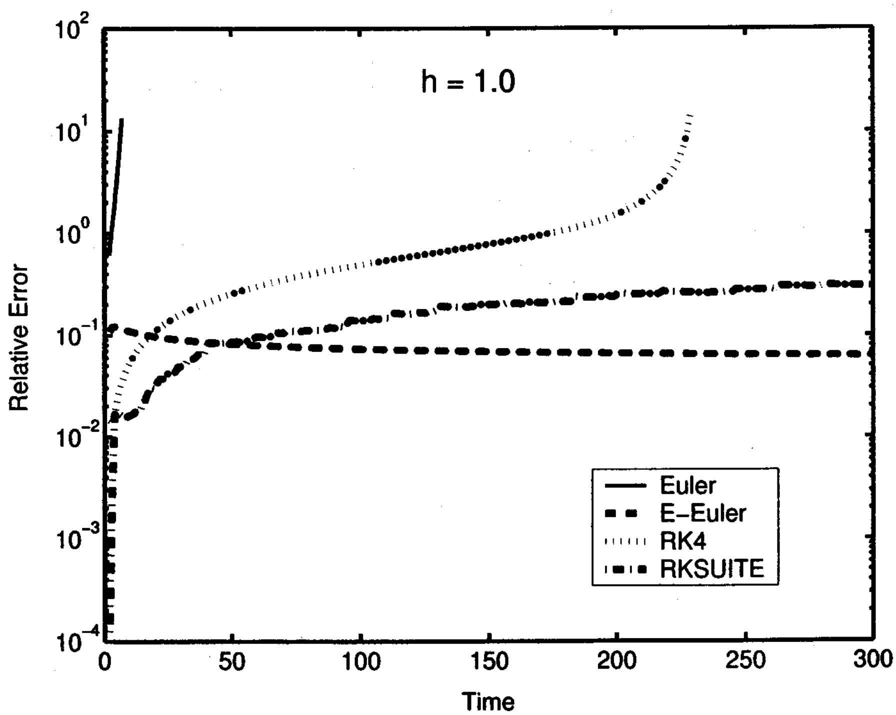



FIGURE 3. Relative errors obtained by Euler, E-Euler, Runge-Kutta and RKSUITE.

Hence, $\tilde{S} = \tilde{J}$ and $P = I$. Using the function $\mathbf{x}(t) = P\mathbf{y}(t)$, the problem becomes

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \frac{2}{t}\cos^2 t + \frac{1}{t}\sin^2 t & \frac{1}{t}\sin t\cos t \\ \frac{1}{t}\sin t\cos t & \frac{2}{t}\sin^2 t + \frac{1}{t}\cos^2 t \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{t^2}\cos t(y_1\cos t + y_2\sin t)^2 - \sin t \\ \frac{1}{t^2}\sin t(y_1\cos t + y_2\sin t)^2 + \cos t \end{pmatrix}.$$

The discrete function of the linear part of $\mathbf{u}(t, \mathbf{y}(t))$ is given by

$$\mathbf{w}(n; h) = \begin{pmatrix} \frac{2}{t_n}\cos^2 t_n + \frac{1}{t_n}\sin^2 t_n & \frac{1}{t_n}\sin t_n\cos t_n \\ \frac{1}{t_n}\sin t_n\cos t_n & \frac{2}{t_n}\sin^2 t_n + \frac{1}{t_n}\cos^2 t_n \end{pmatrix}\begin{pmatrix} y_{1,n} \\ y_{2,n} \end{pmatrix}.$$

Numerical results is obtained for $h = 1.0$, $\alpha = 0.0$, $\beta = 1.0$, $t_0 = 1.0$, $a = -1.0$, $b = 1.0$, $x_0 = x(1.0)$, and $y_0 = y(1.0)$. Figure 3 represents the relative errors obtained by Euler, E-Euler, the classical Runge-Kutta and RKSUITE. The discrete function $\mathbf{w}(n; 1.0)$ is bounded. The Euler method and the classical Runge-Kutta method overflow and can not work for $h = 1.0$. E-Euler process is more efficient than RKSUITE.

3.3. **Concluding Remark.** Based on the examples given in this paper and on many other examples, we have the following information:

When the discrete function $\mathbf{w}(n; h)$ is bounded or gradually decreases to $\epsilon << 1$ as $n \to \infty$, the relative error obtained by E-Euler process converges to a constant or decreases as $n \to \infty$.

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