MULTIPLICATIVE PLANE PARTITIONS

JUN KYO KIM

ABSTRACT. A multiplicative plane partition is a two-dimensional array of positive integers larger than 1 that are nonincreasing both from left to right and top to bottom and whose multiple is a given number n. For a natural number n, let $f_2(n)$ be the number of multiplicative plane partitions of n. In this paper, we prove $f_2(n) \leq n^2$ and a table of them up to 10^5 is provided.

1. Introduction

A multiplicative partition is a liner array of positive integers larger than 1 that are nonincreasing from left to right and whose multiple is a given number n:

$$n = n_1 n_2 \cdots n_s = \prod_{i=1}^s n_i, \qquad n_i \ge n_{i+1}, \quad \text{ and } \quad n_i > 1.$$

For a positive integer n, let f(n) be the number of multiplicative partitions of n. For example, f(12) = 4, since

Canfield et al [2] considered the problem of the maximal order of f(n) and showed that the maximal order is $n \cdot L(n)^{-1+o(1)}$, where

$$L(n) := \exp(\log n \cdot \log_3 n / \log_2 n).$$

Here $\log_k n$ denotes the k-fold iteration of the natural logarithm. The multiplicative partition function f was introduced by Hughes and Shallit [5], who proved that $f(n) \leq 2n^{\sqrt{n}}$ for all positive integer n. Dodd and Mattics [3] improved the inequality so that

$$(1) f(n) \le n$$

for all n. In this paper, we generalize the notion of multiplicative partitions to plane and obtain a corresponding bound.

we can extend the idea of multiplicative partitions to plane as follows. A multiplicative plane partition is a two-dimensional array of positive integers larger than 1 that

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are nonincreasing both from left to right and top to bottom and whose multiple is a given number n:

$$n = \prod_{i,j} n_{i,j}, \qquad n_{i,j} \ge n_{i+1,j}, \quad n_{i,j} \ge n_{i,j+1}, \quad ext{ and } \ n_{i,j} > 1.$$

For example, there are seven multiplicative plane partitions of 12:

For a positive integer n > 1, let $f_2(n)$ be the number of multiplicative plane partitions of n. We abide with the convention $f_2(1) = f(1) = 1$. For a nonnegative integer n, let $P_2(n) = f_2(2^n)$. The function $P_2(n)$ are known as plane partition function. In Section 2, we give an upper bound for $f_2(n)$. The definition of $f_2(n)$ may be extended to $f_j(n)$ in an obvious way.

2. An Upper Bound for $f_2(n)$

In this section, to establish an upper bound on $f_2(n)$, we first define some notations and conventions used in this paper. For a positive integer n, let F(n) be set of all multiplicative partitions of n and $F_2(n)$ be set of all multiplicative plane partitions of n. For a positive integer n, we define a function $\psi_n: F_2(n) \longrightarrow F(n)$ by $\psi_n((a_{i,j})_{i,j}) = (\prod_j a_{i,j})_i$. For a set S, let ||S|| denote the number of elements in S.

Theorem 1. Let n be a positive integer. Then

(2)
$$f_2(n) \le \sum_{(n_1, n_2, \dots, n_r) \in F(n)} \prod_{i=1}^r f(m_i).$$

Proof. Let $\vec{m} = (m_1, m_2, \dots, m_r) \in F(n)$. Then

$$||\psi_n^{-1}(\vec{m})|| \leq \prod_{i=1}^r ||F(m_i)|| = \prod_{i=1}^r f(m_i).$$

Since the function ψ_n is surjective, we have

$$f_2(n) = ||F_2(n)|| = \sum_{(n_1, n_2, \dots, n_r) \in F(n)} ||\psi_n^{-1}(n_1, n_2, \dots, n_r)||$$

$$\leq \sum_{(n_1, n_2, \dots, n_r) \in F(n)} \prod_{i=1}^r f(n_i).$$

Corollary 2. Let n be positive integer. Then

$$(3) f_2(n) \le nf(n).$$

Proof. By Theorem 1 and (1), we have

$$f_{2}(n) \leq \sum_{(n_{1},n_{2},...,n_{r})\in F(n)} \prod_{i=1}^{r} f(n_{i})$$

$$\leq \sum_{(n_{1},n_{2},...,n_{r})\in F(n)} \prod_{i=1}^{r} n_{i}$$

$$= \sum_{(n_{1},n_{2},...,n_{r})\in F(n)} n$$

$$= nf(n).$$

Theorem 3. Let n be positive integer. Then

$$(4) f_2(n) \le n^2.$$

Proof. Immediate from (1) and (3).

3. Remarks and Computations

We say that natural number n is a highly factorable integer on plane if

$$f_2(m) < f_2(n)$$

for all $m, 1 \le m < n$. There is an obvious analogy with the highly factorable numbers n of Canfield et al [2] which satisfy f(m) < f(n) for all $m, 1 \le m \le n$.

In this section, we describe the algorithm used to determine the values displayed in Table 1 and refer some conjectures about the reading asymptotic behavior of $f_2(n)$. Table 1 shows 43 numbers found to be highly factorable.

First, if B_1 and B_2 are elements of each sets $F(m_1)$ and $F(m_2)$, let us write " $B_1 \ge B_2$ " to mean that B_1 is lexicographically larger than B_2 . We may write a multiplicative plane partition π of $F_2(n)$ with the arrays in order

$$\pi = (B_1, B_2, \dots, B_l), \quad B_1 \ge B_2 \ge \dots \ge B_l.$$

Now let l > 1 and $\pi' = (B_2, B_2, \dots, B_l)$. Then $\pi' \in F_2(n/d)$ for some d, d|n and $d \neq 1$. Thus we have the following fact

$$F_2(n) \subset \bigcup_{\substack{d \mid n \ d \neq 1}} F(d) \times F_2(n/d).$$

Table 1. For the convenience of readers, we produce the following table of the highly factorable integers on plane less than 50000.

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Therefore we have

(5)
$$F_2(n) = \{ (B_1, B_2, \dots, B_l) \in \bigcup_{\substack{d \mid n \\ d \neq 1}} F(d) \times F_2(n/d) \mid B_1 \ge B_2 \}.$$

We check that with this definition (5) every partition has a unique parent with one exception, namely the partition whose every block contains one element. If we set $\alpha = \max_{n>1} \frac{\log f_2(n)}{\log n}$ then we showed $\alpha \leq 2$. For a natural number n, let $h(n) = \sum_{(n_1,n_2,\dots,n_r)\in F(n)} \prod_{i=1}^r f(n_i)$. Using the algorithm from [7], the values of h(n) were found for all n less than 10^{13} .

If we set

$$\beta_n = \max_{2 \le m \le n} \frac{\log h(m)}{\log m} \text{ and } \beta = \lim_{n \to \infty} \beta_n,$$

then, for all n for which h(n) were calculated, $\beta < 1.224$. The largest value of β were occurred when n = 2090188800 with h(2090188800) = 252199637381. Based on these data, we propose the following

Conjecture: $f_2(n) < n^{1.224}$ for all n.

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