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MULTIPLICATIVE PLANE PARTITIONS

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ABSTRACT. A multiplicative plane partition is a two-dimensional array of positive integers larger than 1 that are nonincreasing both from left to right and top to bottom and whose multiple is a given number n. For a natural number n, let $f_2(n)$ be the number of multiplicative plane partitions of n. In this paper, we prove $f_2(n) \leq n^2$ and a table of them up to 10^5 is provided.

1. INTRODUCTION

A multiplicative partition is a liner array of positive integers larger than 1 that are nonincreasing from left to right and whose multiple is a given number n:

$$n = n_1 n_2 \cdots n_s = \prod_{i=1}^{s} n_i, \qquad n_i \geq n_{i+1}, \qquad \text{and} \qquad n_i > 1.$$

For a positive integer n, let $f(n)$ be the number of multiplicative partitions of n. For example, $f(12) = 4$, since

$$12, \qquad 6\ \ 2, \qquad 4\ \ 3, \qquad 3\ \ 2\ \ 2.$$

Canfield et al [2] considered the problem of the maximal order of $f(n)$ and showed that the maximal order is $n \cdot L(n)^{-1+o(1)}$, where

$$L(n) := \exp(\log n \cdot \log_3 n / \log_2 n).$$

Here $\log_k n$ denotes the $k-$fold iteration of the natural logarithm. The multiplicative partition function f was introduced by Hughes and Shallit [5], who proved that $f(n) \leq 2n^{\sqrt{n}}$ for all positive integer n. Dodd and Mattics [3] improved the inequality so that

$$f(n) \leq n \tag{1}$$

for all n. In this paper, we generalize the notion of multiplicative partitions to plane and obtain a corresponding bound.

we can extend the idea of multiplicative partitions to plane as follows. A multiplicative plane partition is a two-dimensional array of positive integers larger than 1 that

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are nonincreasing both from left to right and top to bottom and whose multiple is a given number n:

$$n=\prod_{i,j} n_{i,j}, \qquad n_{i,j}\geq n_{i+1,j}, \quad n_{i,j}\geq n_{i,j+1}, \qquad \text{and} \quad n_{i,j}>1.$$

For example, there are seven multiplicative plane partitions of 12:

$$12, \quad 6\ 2, \quad \begin{matrix}6,\\2\end{matrix} \quad 4\ 3, \quad \begin{matrix}4,\\3\end{matrix} \quad 3\ 2\ 2, \quad \begin{matrix}3 & 2,\\2 & \end{matrix} \quad \begin{matrix}3.\\2\\2\end{matrix}$$

For a positive integer $n>1$, let $f_2(n)$ be the number of multiplicative plane partitions of n. We abide with the convention $f_2(1)=f(1)=1$. For a nonnegative integer n, let $P_2(n)=f_2(2^n)$. The function $P_2(n)$ are known as plane partition function. In Section 2, we give an upper bound for $f_2(n)$. The definition of $f_2(n)$ may be extended to $f_j(n)$ in an obvious way.

2. An Upper Bound for $f_2(n)$

In this section, to establish an upper bound on $f_2(n)$, we first define some notations and conventions used in this paper. For a positive integer n, let $F(n)$ be set of all multiplicative partitions of n and $F_2(n)$ be set of all multiplicative plane partitions of n. For a positive integer n, we define a function $\psi_n : F_2(n) \longrightarrow F(n)$ by $\psi_n((a_{i,j})_{i,j}) = (\prod_j a_{i,j})_i$. For a set S, let $||S||$ denote the number of elements in S.

Theorem 1. *Let n be a positive integer. Then*

$$f_2(n) \leq \sum_{(n_1,n_2,\ldots,n_r)\in F(n)} \prod_{i=1}^{r} f(m_i). \tag{2}$$

Proof. Let $\vec{m}=(m_1,m_2,\ldots,m_r)\in F(n)$. Then

$$||\psi_n^{-1}(\vec{m})|| \leq \prod_{i=1}^{r} ||F(m_i)|| = \prod_{i=1}^{r} f(m_i).$$

Since the function ψ_n is surjective, we have

$$\begin{aligned} f_2(n)=||F_2(n)|| &= \sum_{(n_1,n_2,\ldots,n_r)\in F(n)} ||\psi_n^{-1}(n_1,n_2,\ldots,n_r)|| \\ &\leq \sum_{(n_1,n_2,\ldots,n_r)\in F(n)} \prod_{i=1}^{r} f(n_i). \end{aligned}$$

□

Corollary 2. *Let n be positive integer. Then*

$$f_2(n)\leq nf(n). \tag{3}$$

Proof. By Theorem 1 and (1), we have

$$\begin{aligned} f_2(n) &\leq \sum_{(n_1,n_2,\ldots,n_r)\in F(n)} \prod_{i=1}^{r} f(n_i) \\ &\leq \sum_{(n_1,n_2,\ldots,n_r)\in F(n)} \prod_{i=1}^{r} n_i \\ &= \sum_{(n_1,n_2,\ldots,n_r)\in F(n)} n \\ &= nf(n). \end{aligned}$$

□

Theorem 3. *Let n be positive integer. Then*

$$f_2(n) \leq n^2. \tag{4}$$

Proof. Immediate from (1) and (3). □

3. Remarks and Computations

We say that natural number n is a highly factorable integer on plane if

$$f_2(m) < f_2(n)$$

for all m, $1 \leq m < n$. There is an obvious analogy with the highly factorable numbers n of Canfield et al [2] which satisfy $f(m) < f(n)$ for all m, $1 \leq m \leq n$.

In this section, we describe the algorithm used to determine the values displayed in Table 1 and refer some conjectures about the reading asymptotic behavior of $f_2(n)$. Table 1 shows 43 numbers found to be highly factorable.

First, if B_1 and B_2 are elements of each sets $F(m_1)$ and $F(m_2)$, let us write "$B_1 \geq B_2$" to mean that B_1 is lexicographically larger than B_2.We may write a multiplicative plane partition π of $F_2(n)$ with the arrays in order

$$\pi = (B_1, B_2, \ldots, B_l), \quad B_1 \geq B_2 \geq \cdots \geq B_l.$$

Now let $l > 1$ and $\pi' = (B_2, B_2, \ldots, B_l)$. Then $\pi' \in F_2(n/d)$ for some d, $d|n$ and $d \neq 1$. Thus we have the following fact

$$F_2(n) \subset \bigcup_{\substack{d|n \\ d\neq 1}} F(d) \times F_2(n/d).$$

Table 1. For the convenience of readers, we produce the following table of the highly factorable integers on plane less than 50000.

n	$f_2(n)$	$\log(f_2(n))/\log(n)$	decomposition of n
1	1		none
4	3	0.792481	2
8	6	0.861654	3
12	8	0.836829	2 1
16	13	0.92511	4
24	19	0.926491	3 1
32	24	0.916993	5
36	27	0.919721	2 2
48	44	0.977523	4 1
64	48	0.930827	6
72	65	0.976084	3 2
96	94	0.995387	5 1
144	158	1.01867	4 2
192	195	1.00295	6 1
240	216	0.980776	4 1 1
288	345	1.03189	5 2
384	387	1.00131	7 1
432	443	1.00414	4 3
480	488	1.00268	5 1 1
576	744	1.04027	6 2
720	820	1.01977	4 2 1
864	1027	1.02556	5 3
960	1059	1.01429	6 1 1
1152	1516	1.03895	7 2
1440	1948	1.04155	5 2 1
1728	2284	1.03742	6 3
2160	2569	1.02259	4 3 1
2304	3038	1.03572	8 2
2880	4407	1.05341	6 2 1
3456	4845	1.04146	7 3
4320	6359	1.04619	5 3 1
5184	6489	1.02625	6 4
5760	9516	1.05798	7 2 1
6912	9981	1.04156	8 3
8640	14893	1.06007	6 3 1
11520	19871	1.0583	8 2 1
13824	19945	1.03845	9 3
17280	33220	1.06699	7 3 1
23040	40205	1.05543	9 2 1
25920	44806	1.05386	6 4 1
30240	45559	1.03973	5 3 1 1
34560	71386	1.06941	8 3 1
46080	79334	1.05059	10 2 1
51840	102766	1.06303	7 4 1
60480	111836	1.05583	6 3 1 1
69120	148411	1.06857	9 3 1
80640	151687	1.05592	8 2 1 1
86400	162000	1.0553	7 3 2

Therefore we have

$$F_2(n) = \{(B_1, B_2, \ldots, B_l) \in \bigcup_{\substack{d|n \\ d\neq 1}} F(d) \times F_2(n/d) \mid B_1 \geq B_2\}. \tag{5}$$

We check that with this definition (5) every partition has a unique parent with one exception, namely the partition whose every block contains one element. If we set $\alpha = \max_{n>1} \frac{\log f_2(n)}{\log n}$ then we showed $\alpha \leq 2$. For a natural number n, let $h(n) = \sum_{(n_1,n_2,\ldots,n_r)\in F(n)} \prod_{i=1}^{r} f(n_i)$. Using the algorithm from [7], the values of $h(n)$ were found for all n less than 10^{13}.

If we set

$$\beta_n = \max_{2\leq m\leq n} \frac{\log h(m)}{\log m} \quad \text{and} \quad \beta = \lim_{n\to\infty} \beta_n,$$

then, for all n for which $h(n)$ were calculated, $\beta < 1.224$. The largest value of β were occurred when $n = 2090188800$ with $h(2090188800) = 252199637381$. Based on these data, we propose the following

Conjecture: $f_2(n) < n^{1.224}$ for all n.

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