Kangweon-Kyungki Math. Jour. **11** (2003), No. 2, pp. 169–175

G-FUZZY EQUIVALENCE RELATIONS GENERATED BY FUZZY RELATIONS

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Abstract. We find a G-fuzzy equivalence relation generated by the union of two G-fuzzy equivalence relations in a set, find a G-fuzzy equivalence relation generated by a fuzzy relation in a set, and find sufficient conditions for the composition $\mu \circ \nu$ of two G-fuzzy equivalence relations μ and ν to be a G-fuzzy equivalence relation generated by $\mu \cup \nu$.

1. Introduction

The concept of a fuzzy relation was first proposed by Zadeh ([6]). Subsequently, Goguen ([1]) and Sanchez ([5]) studied fuzzy relations in various contexts. In [4] Nemitz discussed fuzzy equivalence relations, fuzzy functions as fuzzy relations, and fuzzy partitions. Murali ([3]) developed some properties of fuzzy equivalence relations and certain lattice theoretic properties of fuzzy equivalence relations. Gupta et al. ([2]) proposed a generalized definition of a fuzzy equivalence relation on a set, which we call G-fuzzy equivalence relation in this paper, and developed some properties of that relation. The present work has been started as a continuation of these studies.

In section 2 we develop some basic properties of fuzzy relations, find a G-fuzzy equivalence relation generated by the union of two G-fuzzy equivalence relations in a set, find a G-fuzzy equivalence relation generated by a fuzzy relation in a set, and show that if μ and ν are G-fuzzy equivalence relations in a set such that $\mu \circ \nu = \nu \circ \mu$, $\inf_{t \in X} \mu(t,t) \geq \nu(x,y)$, and $\inf_{t \in X} \nu(t,t) \geq \mu(x,y)$ for all $x \neq y \in X$, then $\mu \circ \nu$ is a G-fuzzy equivalence relation generated by $\mu \cup \nu$.

Received July 24, 2003.

2000 Mathematics Subject Classification: 03E72.

Key words and phrases: fuzzy relation, G-fuzzy equivalence relation .

This paper was supported by the Natural Science Research Institute of Seoul Women's University, 2002

2. Fuzzy equivalence relation

DEFINITION 2.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in B$, $B(x)$ is called a *membership grade* of x in B.

The standard definition of a fuzzy reflexive relation μ in a set X demands $\mu(x,x) = 1$. Gupta et al. ([2]) weakened this definition as follows.

DEFINITION 2.2. A *fuzzy relation* μ in a set X is a fuzzy subset of $X \times X$. μ is *G-reflexive* in X if $\mu(x,x) > 0$ and $\mu(x,y) \leq \inf\limits_{t \in X} \mu(t,t)$ for all $x \neq y$ in X. μ is *symmetric* in X if $\mu(x,y) = \mu(y,x)$ for all x,y in X. The composition $\lambda \circ \mu$ of two fuzzy relations λ, μ in X is the fuzzy subset of $X \times X$ defined by

$$(\lambda \circ \mu)(x,y) = \sup_{z \in X} \min(\lambda(x,z), \mu(z,y)).$$

A fuzzy relation μ in X is *transitive* in X if $\mu \circ \mu \subseteq \mu$. A fuzzy relation μ in X is called *G-fuzzy equivalence relation* if μ is G-reflexive, symmetric, and transitive.

PROPOSITION 2.3. *Let $\mathcal{F}_X$ be the set of all fuzzy relations in a set X. Then $\mathcal{F}_X$ is a monoid under the operation of composition $\circ$.*

Proof. Clearly $\circ$ is a binary operation. It is well known that $\circ$ is associative (see Proposition 2.3 of [3]). Let θ be a fuzzy relation such that $\theta(x,x) = 1$ and $\theta(x,y) = 0$ if $x \neq y$. Then $(\mu \circ \theta)(x,y) = \sup\limits_{z \in X} \min(\mu(x,z), \theta(z,y)) = \mu(x,y)$. Similarly we may show $(\theta \circ \mu)(x,y) = \mu(x,y)$. Hence $\mathcal{F}_X$ is a monoid. $\square$

It is easy to see that a G-fuzzy equivalence relation is an idempotent element of $\mathcal{F}_X$.

DEFINITION 2.4. Let μ be a fuzzy relation in a set X. μ^{-1} is defined as a fuzzy relation in X by $\mu^{-1}(x,y) = \mu(y,x)$.

It is easy to see that $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1}$ for fuzzy relations μ and ν.

PROPOSITION 2.5. *Let $\mathcal{F}_X$ be a monoid of all fuzzy relations in X and let $\phi : F_X \to F_X$ be a map defined by $\phi(\mu) = \mu^{-1}$. Then ϕ is an antiautomorphism and $\phi(\mu^{-1}) = (\phi(\mu))^{-1} = \mu$.*

Proof. Since $(\mu^{-1})^{-1}(x, y) = \mu^{-1}(y, x) = \mu(x, y)$ for all $x, y \in X$, $\phi(\mu^{-1}) = (\mu^{-1})^{-1} = \mu = (\phi(\mu))^{-1}$. Since $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1}$, $\phi(\mu \circ \nu) = (\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1} = \phi(\nu) \circ \phi(\mu)$. □

PROPOSITION 2.6. *Let μ and each ν_i be fuzzy relations in a set X for all $i \in I$. Then $\mu \circ (\bigcup_{i\in I} \nu_i) = \bigcup_{i\in I}(\mu \circ \nu_i)$, $(\bigcup_{i\in I} \nu_i) \circ \mu = \bigcup_{i\in I}(\nu_i \circ \mu)$, $\mu \circ (\bigcap_{i\in I} \nu_i) \subseteq \bigcap_{i\in I}(\mu \circ \nu_i)$, and $(\bigcap_{i\in I} \nu_i) \circ \mu \subseteq \bigcap_{i\in I}(\nu_i \circ \mu)$.*

Proof.

$$\begin{aligned}
[\mu \circ (\bigcup_{i\in I} \nu_i)](x, y) &= \sup_{z\in X} \min[\mu(x, z), (\bigcup_{i\in I} \nu_i)(z, y)] \\
&= \sup_{z\in X} \min[\mu(x, z), \sup_{i\in I} \nu_i(z, y)] \\
&= \sup_{z\in X} \sup_{i\in I} \min[\mu(x, z), \nu_i(z, y)] \\
&= \sup_{i\in I} \sup_{z\in X} \min[\mu(x, z), \nu_i(z, y)] \\
&= (\bigcup_{i\in I} \mu \circ \nu_i)(x, y).
\end{aligned}$$

Similarly we may prove the remaining things. □

PROPOSITION 2.7. *Let μ and ν be G-fuzzy equivalence relations in a set X. Then $\mu \cap \nu$ is a G-fuzzy equivalence relation.*

Proof. $(\mu \cap \nu)(x, x) = \min(\mu(x, x), \nu(x, x)) > 0$.

$$\begin{aligned}
\inf_{t\in X} (\mu \cap \nu)(t, t) &= \inf_{t\in X} \min(\mu(t, t), \nu(t, t)) \\
&= \min (\inf_{t\in X} \mu(t, t), \inf_{t\in X} \nu(t, t)) \\
&\geq \min (\mu(x, y), \nu(x, y)) = (\mu \cap \nu)(x, y)
\end{aligned}$$

for all $x \neq y$ in X. That is, $\mu \cap \nu$ is G-reflesive. $(\mu \cap \nu)(x, y) = \min(\mu(x, y), \nu(x, y)) = \min(\mu(y, x), \nu(y, x)) = (\mu \cap \nu)(y, x)$. By Proposition 2.6, $[(\mu \cap \nu) \circ (\mu \cap \nu)] \subseteq [\mu \circ (\mu \cap \nu)] \cap [\nu \circ (\mu \cap \nu)] \subseteq [(\mu \circ \mu) \cap (\mu \circ \nu)] \cap [(\nu \circ \mu) \cap (\nu \circ \nu)] \subseteq [\mu \cap (\mu \circ \nu)] \cap [(\nu \circ \mu) \cap \nu] \subseteq \mu \cap \nu$. □

It is easy to see that even though μ and ν are G-fuzzy equivalence relations, $\mu \cup \nu$ is not necessarily a G-fuzzy equivalence relation. We find a G-fuzzy equivalence relation generated by $\mu \cup \nu$ in the following theorem.

THEOREM 2.8. *Let μ and ν be G-fuzzy equivalence relations in a set X. The G-fuzzy equivalence relation generated by $\mu \cup \nu$ is $\cup_{n=1}^{\infty}(\mu \cup \nu)^n = (\mu \cup \nu) \cup [(\mu \cup \nu) \circ (\mu \cup \nu)] \cup \dots$.*

Proof. Clearly $(\mu \cup \nu)(x, x) > 0$.

$$\begin{aligned} \inf_{t \in X} (\mu \cup \nu)(t,t) &= \inf_{t \in X} \max(\mu(t,t), \nu(t,t)) \\ &= \max\,(\inf_{t \in X} \mu(t,t), \inf_{t \in X} \nu(t,t)) \\ &\geq \max\,(\mu(x,y), \nu(x,y)) = (\mu \cup \nu)(x,y) \end{aligned}$$

for all $x \neq y$ in X. That is, $\mu \cup \nu$ is G-reflexive. $[(\mu \cup \nu) \circ (\mu \cup \nu)](x,x) = \sup_{z \in X} \min[(\mu \cup \nu)(x,z), (\mu \cup \nu)(z,x)] \geq \min[(\mu \cup \nu)(x,x), (\mu \cup \nu)(x,x)] > 0$. $\inf_{t \in X} [(\mu \cup \nu) \circ (\mu \cup \nu)](t,t) = \inf_{t \in X} \sup_{z \in X} \min[(\mu \cup \nu)(t,z), (\mu \cup \nu)(z,t)] \geq \inf_{t \in X} \min[(\mu \cup \nu)(t,t), (\mu \cup \nu)(t,t)] = \inf_{t \in X} (\mu \cup \nu)(t,t) \geq \sup_{z \in X} \min[(\mu \cup \nu)(x,z), (\mu \cup \nu)(z,y)] = ((\mu \cup \nu) \circ (\mu \cup \nu))(x,y)$. That is, $(\mu \cup \nu) \circ (\mu \cup \nu)$ is G-reflexive. Similarly $(\mu \cup \nu)^n$ is G-reflexive for $n = 3, 4, \dots$. $\inf_{t \in X} [\cup_{n=1}^{\infty}(\mu \cup \nu)^n](t,t) = \inf_{t \in X} \sup[(\mu \cup \nu)(t,t), ((\mu \cup \nu) \circ (\mu \cup \nu))(t,t), \dots] = \sup\,[\inf_{t \in X} (\mu \cup \nu)(t,t), \inf_{t \in X} ((\mu \cup \nu) \circ (\mu \cup \nu))(t,t), \dots] \geq \sup[(\mu \cup \nu)(x,y), ((\mu \cup \nu) \circ (\mu \cup \nu))(x,y), \dots] = [\cup_{n=1}^{\infty}(\mu \cup \nu)^n](x,y)$. Clearly $[\cup_{n=1}^{\infty}(\mu \cup \nu)^n](x,x) > 0$. Thus $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is G-reflexive. Clearly $\mu \cup \nu$ is symmetric. $[(\mu \cup \nu) \circ (\mu \cup \nu)](x,y) = \sup_{z \in X} \min[(\mu \cup \nu)(x,z), (\mu \cup \nu)(z,y)] = \sup_{z \in X} \min[(\mu \cup \nu)(y,z), (\mu \cup \nu)(z,x)] = [(\mu \cup \nu) \circ (\mu \cup \nu)](y,x)$. That is, $(\mu \cup \nu) \circ (\mu \cup \nu)$ is symmetric. Similarly we may show $(\mu \cup \nu)^n$ is symmetric for $n = 3, 4, \dots$. $[\cup_{n=1}^{\infty}(\mu \cup \nu)^n](x,y) = \sup[(\mu \cup \nu)(x,y), ((\mu \cup \nu) \circ (\mu \cup \nu))(x,y), \dots] = \sup[(\mu \cup \nu)(y,x), ((\mu \cup \nu) \circ (\mu \cup \nu))(y,x), \dots] = [\cup_{n=1}^{\infty}(\mu \cup \nu)^n](y,x)$. That is, $\cup_{n=1}^{\infty}(\mu \cup \nu)^n$ is symmetric. Let $\mu^* = \cup_{n=1}^{\infty}(\mu \cup \nu)^n$ and let $\mu_1 = \mu \cup \nu$. By Proposition 2.6, $\mu^* \circ \mu^* = \mu^* \circ [\mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cup \dots] = [\mu^* \circ \mu_1] \cup [\mu^* \circ (\mu_1 \circ \mu_1)] \cup [\mu^* \circ (\mu_1 \circ \mu_1 \circ \mu_1)] \cup \dots = [(\mu_1 \circ \mu_1) \cup ((\mu_1 \circ \mu_1) \circ \mu_1) \cup \dots] \cup$

$[(\mu_1 \circ (\mu_1 \circ \mu_1)) \cup ((\mu_1 \circ \mu_1) \circ (\mu_1 \circ \mu_1)) \cup \ldots] \cup \cdots = [(\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cup \ldots] \cup [(\mu_1 \circ \mu_1 \circ \mu_1) \cup \ldots] \cup \cdots \subseteq \mu^*$. That is, $\mu^* = \cup_{n=1}^{\infty} (\mu \cup \nu)^n$ is transitive. Hence $\cup_{n=1}^{\infty} (\mu \cup \nu)^n$ is a G-fuzzy equivalence relation. Let λ be a G-fuzzy equivalence relations in a set X containing $\mu \cup \nu$. Then $\cup_{n=1}^{\infty} (\mu \cup \nu)^n \subseteq \cup_{n=1}^{\infty} \lambda^n = \lambda \cup (\lambda \circ \lambda) \cup (\lambda \circ \lambda \circ \lambda) \cup \cdots \subseteq \lambda \cup \lambda \cup \cdots = \lambda$. That is, $\cup_{n=1}^{\infty} (\mu \cup \nu)^n$ is contained in every G-fuzzy equivalence relation in X containing $\mu \cup \nu$. Thus $\cup_{n=1}^{\infty} (\mu \cup \nu)^n$ is a G-fuzzy equivalence relation generated by $\mu \cup \nu$. $\square$

THEOREM 2.9. *Let μ be a fuzzy relation in a set X. Then G-fuzzy equivalence relation in X generated by μ is $\mu^* = \cup_{n=1}^{\infty} \mu_1^n = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cup \ldots$, where $\mu_1 = \mu \cup \mu^{-1} \cup \theta$ and θ is a fuzzy relation in X such that $\theta(x,x) > 0$, $\theta = \theta^{-1}$, $\theta(x,y) \leq \mu(x,y)$, and $\max[\mu(x,y), \theta(x,y)] \leq \inf_{t \in X} \theta(t,t)$ for all $x \neq y$ in X.*

Proof. $(\mu \cup \mu^{-1} \cup \theta)(x,x) = \max[\mu(x,x), \mu^{-1}(x,x), \theta(x,x)] > 0$.

$$\begin{aligned}\inf_{t \in X} (\mu \cup \mu^{-1} \cup \theta)(t,t) &= \inf_{t \in X} \max[\mu(t,t),\ \mu^{-1}(t,t),\ \theta(t,t)] \\ &\geq \inf_{t \in X} \theta(t,t) \geq \max[\mu(x,y),\ \mu^{-1}(x,y),\ \theta(x,y)] \\ &= (\mu \cup \mu^{-1} \cup \theta)(x,y).\end{aligned}$$

Thus $\mu_1 = \mu \cup \mu^{-1} \cup \theta$ is G-reflexive. By the same way as shown in Theorem 2.8, we may show $\mu^* = \cup_{n=1}^{\infty} \mu_1^n$ is G-reflexive.

$$\begin{aligned}\mu_1(x,y) = (\mu \cup \mu^{-1} \cup \theta)(x,y) &= \max[\mu(x,y), \mu^{-1}(x,y), \theta^{-1}(x,y)] \\ &= \max[\mu^{-1}(y,x), \mu(y,x), \theta(y,x)] \\ &= (\mu \cup \mu^{-1} \cup \theta)(y,x) = \mu_1(y,x).\end{aligned}$$

Thus μ_1 is a symmetric. By the same way as shown in Theorem 2.8, we may show $\mu^* = \cup_{n=1}^{\infty} \mu_1^n$ is symmetric and transitive. Hence μ^* is a G-fuzzy equivalence relation containing μ. Let ν be a G-fuzzy equivalence relation containing μ. Then $\mu(x,y) \leq \nu(x,y)$, $\mu^{-1}(x,y) = \mu(y,x) \leq \nu(y,x) = \nu(x,y)$, and $\theta(x,y) \leq \mu(x,y) \leq \nu(x,y)$. Thus $\mu_1 = (\mu \cup \mu^{-1} \cup \theta) \subseteq \nu$. $(\mu_1 \circ \mu_1)(x,y) = \sup_{z \in X} \min[\mu_1(x,z), \mu_1(z,y)] \leq \sup_{z \in X} \min[\nu(x,z), \nu(z,y)] = (\nu \circ \nu)(x,y)$. Since ν is transitive, $\mu_1 \circ \mu_1 \subseteq \nu \circ \nu \subseteq \nu$. Similarly we may show $\mu_1^n \subseteq \nu$ for $n = 3, \ldots$. Thus $\mu^* = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cdots \subseteq \nu$. $\square$

THEOREM 2.10. *Let μ and ν be G-fuzzy equivalence relations in a set X such that $\inf_{t\in X} \mu(t,t) \geq \nu(x,y)$ and $\inf_{t\in X} \nu(t,t) \geq \mu(x,y)$ for all $x \neq y \in X$. If $\mu\circ\nu = \nu\circ\mu$, then $\mu\circ\nu$ is a G-fuzzy equivalence relation in X generated by $\mu\cup\nu$.*

Proof.

$$\begin{aligned}(\mu\circ\nu)(x,x) &= \sup_{z\in X} \min[\mu(x,z),\nu(z,x)] \\ &\geq \min(\mu(x,x),\nu(x,x)) > 0.\end{aligned}$$

Since $\inf_{t\in X} \mu(t,t) \geq \nu(x,y)$ and $\inf_{t\in X} \nu(t,t) \geq \mu(x,y)$ for all $x \neq y \in X$,

$$\begin{aligned}\inf_{t\in X} (\mu\circ\nu)(t,t) &= \inf_{t\in X} \sup_{z\in X} \min[\mu(t,z),\nu(z,t)] \\ &\geq \inf_{t\in X} \min[\mu(t,t),\nu(t,t)] \geq \min[\mu(x,z),\nu(z,y)]\end{aligned}$$

for all $z \in X$. Thus $\inf_{t\in X} (\mu\circ\nu)(t,t) \geq \sup_{z\in X} \min[\mu(x,z),\nu(z,y)] = (\mu\circ\nu)(x,y)$. That is, $\mu\circ\nu$ is G-reflexive. Since μ and ν are symmetric, $(\mu\circ\nu)^{-1} = \nu^{-1}\circ\mu^{-1} = \nu\circ\mu = \mu\circ\nu$. Thus $\mu\circ\nu$ is symmetric. Since μ and ν are transitive and the operation $\circ$ is associative, $(\mu\circ\nu)\circ(\mu\circ\nu) = \mu\circ(\nu\circ\mu)\circ\nu = \mu\circ(\mu\circ\nu)\circ\nu = (\mu\circ\mu)\circ(\nu\circ\nu) \subseteq \mu\circ\nu$. Hence $\mu\circ\nu$ is a G-fuzzy equivalence relation. Since $\nu(y,y) \geq \mu(x,y)$, $(\mu\circ\nu)(x,y) = \sup_{z\in X} \min[\mu(x,z),\nu(z,y)] \geq \min(\mu(x,y),\nu(y,y)) = \mu(x,y)$. Since $\mu(x,x) \geq \nu(x,y)$, $(\mu\circ\nu)(x,y) = \sup_{z\in X} \min[\mu(x,z),\nu(z,y)] \geq \min(\mu(x,x),\nu(x,y)) = \nu(x,y)$. Thus $(\mu\circ\nu)(x,y) \geq \max(\mu(x,y),\nu(x,y)) = (\mu\cup\nu)(x,y)$. That is, $\mu\cup\nu \subseteq \mu\circ\nu$. Let λ be a G-fuzzy equivalence relation in X containing $\mu\cup\nu$. Since λ is transitive, $\mu\circ\nu \subseteq (\mu\cup\nu)\circ(\mu\cup\nu) \subseteq \lambda\circ\lambda \subseteq \lambda$. Thus $\mu\circ\nu$ is a G-fuzzy equivalence relation generated by $\mu\cup\nu$. □

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