Kangweon-Kyungki Math. Jour. **11** (2003), No. 2, pp. 127–136

WEAK* SMOOTH COMPACTNESS IN SMOOTH TOPOLOGICAL SPACES

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Abstract. In this paper we obtain some properties of the weak smooth α-closure and weak smooth α-interior of a fuzzy set in smooth topological spaces and introduce the concepts of several types of weak* smooth compactness in smooth topological spaces and investigate some of their properties.

1. Introduction

Badard [1] introduced the concept of a smooth topological space which is a generalization of Chang's fuzzy topological space [2]. Many mathematical structures in smooth topological spaces were introduced and studied. In particular, Gayyar, Kerre and Ramadan [5] and Demirci [3, 4] introduced the concepts of smooth closure and smooth interior of a fuzzy set and several types of compactness in smooth topological spaces and obtained some of their properties. In [6] we introduced the concepts of smooth α-closure and smooth α-interior of a fuzzy set which are generalizations of smooth closure and smooth interior of a fuzzy set defined in [3] and also introduced several types of α-compactness in smooth topological spaces and obtained some of their properties. In [7] we introduced the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set and investigated some of their properties.

In this paper we obtain some properties of the weak smooth α-closure and weak smooth α-interior of a fuzzy set in smooth topological spaces and introduce the concepts of several types of weak* smooth compactness in smooth topological spaces and investigate some of their properties.

Received July 7, 2003.

2000 Mathematics Subject Classification: 54A40, 03E72.

Key words and phrases: fuzzy sets, smooth topology, weak smooth α-closure, weak smooth α-interior, weak* smooth compactness.

2. Preliminaries

Let X be a set and $I = [0, 1]$ be the unit interval of the real line. I^X will denote the set of all fuzzy sets of X. 0_X and 1_X will denote the characteristic functions of ϕ and X, respectively.

A smooth topological space (s.t.s.) [8] is an ordered pair (X, τ), where X is a non-empty set and $\tau : I^X \to I$ is a mapping satisfying the following conditions:

(O1) $\tau(0_X) = \tau(1_X) = 1$;

(O2) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(O3) for every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \tau(A_i)$.
Then the mapping $\tau : I^X \to I$ is called a smooth topology on X. The number $\tau(A)$ is called the degree of openness of A.

A mapping $\tau^* : I^X \to I$ is called a smooth cotopology [8] iff the following three conditions are satisfied:

(C1) $\tau^*(0_X) = \tau^*(1_X) = 1$;

(C2) $\forall A, B \in I^X,\ \tau^*(A \cup B) \geq \tau^*(A) \wedge \tau^*(B)$;

(C3) for every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau^*(\cap_{i\in J} A_i) \geq \wedge_{i\in J} \tau^*(A_i)$.

If τ is a smooth topology on X, then the mapping $\tau^* : I^X \to I$, defined by $\tau^*(A) = \tau(A^c)$ where A^c denotes the complement of A, is a smooth cotopology on X. Conversely, if τ^* is a smooth cotopology on X, then the mapping $\tau : I^X \to I$, defined by $\tau(A) = \tau^*(A^c)$, is a smooth topology on X [8].

Demirci [3] introduced the concepts of smooth closure and smooth interior in smooth topological spaces as follows:

Let (X, τ) be a s.t.s. and $A \in I^X$. Then the τ-smooth closure (resp., τ-smooth interior) of A, denoted by $\bar{A}$ (resp., A^o), is defined by $\bar{A} = \cap\{K \in I^X : \tau^*(K) > 0, A \subseteq K\}$ (resp., $A^o = \cup\{K \in I^X : \tau(K) > 0, K \subseteq A\}$). Demirci [4] defined the families $W(\tau) = \{A \in I^X : A = A^o\}$ and $W^*(\tau) = \{A \in I^X : A = \overline{A}\}$, where (X, τ) is a s.t.s. Note that $A \in W(\tau) \Leftrightarrow A^c \in W^*(\tau)$.

Let (X, τ) and (Y, σ) be two smooth topological spaces. A function $f : X \to Y$ is called smooth continuous with respect to τ and σ [8] iff $\tau(f^{-1}(A)) \geq \sigma(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is called weakly smooth continuous with respect to τ and σ [8] iff $\sigma(A) > 0 \Rightarrow \tau(f^{-1}(A)) > 0$ for every $A \in I^Y$. In this paper, a weakly smooth continuous function with respect to τ and σ is called a quasi-smooth continuous function with respect to τ and σ.

A function $f : X \to Y$ is smooth continuous with respect to τ and σ iff $\tau^*(f^{-1}(A)) \geq \sigma^*(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is weakly smooth continuous with respect to τ and σ iff $\sigma^*(A) > 0 \Rightarrow \tau^*(f^{-1}(A)) > 0$ for every $A \in I^Y$ [8].

A function $f : X \to Y$ is called smooth open (resp., smooth closed) with respect to τ and σ [8] iff $\tau(A) \leq \sigma(f(A))$ (resp., $\tau^*(A) \leq \sigma^*(f(A))$) for every $A \in I^X$.

A function $f : X \to Y$ is called smooth preserving (resp., strict smooth preserving) with respect to τ and σ [5] iff $\sigma(A) \geq \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) \geq \tau(f^{-1}(B))$ (resp., $\sigma(A) > \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) > \tau(f^{-1}(B))$) for every $A, B \in I^Y$.

If $f : X \to Y$ is a smooth preserving function (resp., a strict smooth preserving function) with respect to τ and σ, then $\sigma^*(A) \geq \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) \geq \tau^*(f^{-1}(B))$ (resp., $\sigma^*(A) > \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) > \tau^*(f^{-1}(B))$) for every $A, B \in I^Y$ [5].

A function $f : X \to Y$ is called smooth open preserving (resp., strict smooth open preserving) with respect to τ and σ [5] iff $\tau(A) \geq \tau(B) \Rightarrow \sigma(f(A)) \geq \sigma(f(B))$ (resp., $\tau(A) > \tau(B) \Rightarrow \sigma(f(A)) > \sigma(f(B))$) for every $A, B \in I^X$.

Let (X, τ) be a s.t.s., $\alpha \in [0, 1)$ and $A \in I^X$. The τ-smooth α-closure (resp., τ-smooth α-interior) of A, denoted by $\overline{A}_\alpha$ (resp., A^o_α), is defined by $\overline{A}_\alpha = \cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A), A \subseteq K\}$ (resp., $A^o_\alpha = \cup\{K \in I^X : \tau(K) > \alpha\tau(A), K \subseteq A\}$) [6]. In [7] we defined the families $W_\alpha(\tau) = \{A \in I^X : A = A^o_\alpha\}$ and $W^*_\alpha(\tau) = \{A \in I^X : A = \overline{A}_\alpha\}$, where (X, τ) is a s.t.s. Note that $A \in W_\alpha(\tau) \Leftrightarrow A^c \in W^*_\alpha(\tau)$.

3. weak smooth α-closure and weak smooth α-interior

In this section, we investigate some properties of the weak smooth α-closure and weak smooth α-interior of a fuzzy set in smooth topological spaces.

Definition 3.1[7]. Let (X, τ) be a s.t.s., $\alpha \in [0, 1)$ and $A \in I^X$. The weak τ-smooth α-closure (resp., weak τ-smooth α-interior) of A, denoted by $wcl_\alpha(A)$ (resp., $wint_\alpha(A)$), is defined by $wcl_\alpha(A) = \cap\{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\}$ (resp., $wint_\alpha(A) = \cup\{K \in I^X : K \in$

$W_\alpha(\tau), K \subseteq A\})$.

THEOREM 3.2. *Let* (X,τ) *be a s.t.s.,* $\alpha \in [0,1)$ *and* $A \in I^X$. *Then*
(a) $A \subseteq wcl_\alpha(A) \subseteq \overline{A} \subseteq \overline{A}_\alpha$,
(b) $A^o_\alpha \subseteq A^o \subseteq wint_\alpha(A) \subseteq A$.

Proof. (a) Let $K \in I^X$ and $A \subseteq K$. Then $\tau^*(K) > \alpha\tau^*(A) \Rightarrow \tau^*(K) > 0$ and $\tau^*(K) > 0 \Rightarrow K = \overline{K}_\alpha$, i.e., $K \in W^*_\alpha(\tau)$ by Theorem 3.6[6]. From the definitions of $\overline{A}_\alpha$, $\overline{A}$ and $wcl_\alpha(A)$ we have $A \subseteq wcl_\alpha(A) \subseteq \overline{A} \subseteq \overline{A}_\alpha$.

(b) Let $K \in I^X$ and $K \subseteq A$. Then $\tau(K) > \alpha\tau(A) \Rightarrow \tau(K) > 0$ and $\tau(K) > 0 \Rightarrow K = K^o_\alpha$, i.e., $K \in W_\alpha(\tau)$ by Theorem 3.6[6]. From the definition of A^o_α, A^o and $wint_\alpha(A)$ we have $A^o_\alpha \subseteq A^o \subseteq wint_\alpha(A) \subseteq A$. □

THEOREM 3.3. *Let* (X,τ) *be a s.t.s.,* $\alpha \in [0,1)$ *and* $A, B \in I^X$. *Then*
(a) $A \subseteq B \Rightarrow wcl_\alpha(A) \subseteq wcl_\alpha(B)$,
(b) $A \subseteq B \Rightarrow wint_\alpha(A) \subseteq wint_\alpha(B)$,
(c) $(wcl_\alpha(A))^c = wint_\alpha(A^c)$,
(d) $wcl_\alpha(A) = (wint_\alpha(A^c))^c$,
(e) $(wint_\alpha(A))^c = wcl_\alpha(A^c)$,
(f) $wint_\alpha(A) = (wcl_\alpha(A^c))^c$.

Proof. (a) and (b) follow directly from Definition 3.2.

(c) From Definition 3.2 we have

$$\begin{aligned}(wcl_\alpha(A))^c &= (\cap\{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\})^c \\ &= \cup\{K^c : K \in I^X, K^c \in W_\alpha(\tau), K^c \subseteq A^c\} \\ &= \cup\{U \in I^X : U \in W_\alpha(\tau), U \subseteq A^c\} \\ &= wint_\alpha(A^c).\end{aligned}$$

(d), (e) and (f) can be easily obtained from (c). □

DEFINITION 3.4[4]. Let (X,τ) and (Y,σ) be two smooth topological spaces. A function $f : X \to Y$ is called weak smooth continuous with respect to τ and σ iff $A \in W(\sigma) \Rightarrow f^{-1}(A) \in W(\tau)$ for every $A \in I^Y$.

Let (X,τ) and (Y,σ) be two smooth topological spaces. A function $f : X \to Y$ is weak smooth continuous with respect to τ and σ iff $A \in W^*(\sigma) \Rightarrow f^{-1}(A) \in W^*(\tau)$ for every $A \in I^Y$ [4].

THEOREM 3.5. *Let (X, τ) and (Y, σ) be two smooth topological spaces. If a function $f : X \to Y$ is quasi-smooth continuous with respect to τ and σ, then $f : X \to Y$ is weak smooth continuous with respect to τ and σ.*

Proof. Let $f : X \to Y$ be a quasi-smooth continuous function with respect to τ and σ. Then by Proposition 3.5[3] $f^{-1}(A^o) \subseteq (f^{-1}(A))^o$ for every $A \in I^Y$. Let $A \in W(\sigma)$, i.e., $A = A^o$. Then $f^{-1}(A) = f^{-1}(A^o) \subseteq (f^{-1}(A))^o$. From the definition of smooth interior we have $(f^{-1}(A))^o \subseteq f^{-1}(A)$. Hence $f^{-1}(A) = (f^{-1}(A))^o$, i.e., $f^{-1}(A) \in W(\tau)$. Therefore $f : X \to Y$ is weak smooth continuous with respect to τ and σ. □

DEFINITION 3.6. Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. A function $f : X \to Y$ is called weak smooth α-continuous with respect to τ and σ iff $A \in W_\alpha(\sigma) \Rightarrow f^{-1}(A) \in W_\alpha(\tau)$ for every $A \in I^Y$.

THEOREM 3.7. *Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. If a function $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, then*

(a) $f(wcl_\alpha(A)) \subseteq wcl_\alpha(f(A))$ for every $A \in I^X$,

(b) $wcl_\alpha(f^{-1}(A)) \subseteq f^{-1}(wcl_\alpha(A))$ for every $A \in I^Y$,

(c) $f^{-1}(wint_\alpha(A)) \subseteq wint_\alpha(f^{-1}(A))$ for every $A \in I^Y$.

Proof. (a) For every $A \in I^X$, we have

$$\begin{aligned}
&f^{-1}(wcl_\alpha(f(A))) \\
&\quad = f^{-1}(\cap\{U \in I^Y : U \in W^*_\alpha(\sigma), f(A) \subseteq U\}) \\
&\quad \supseteq f^{-1}(\cap\{U \in I^Y : f^{-1}(U) \in W^*_\alpha(\tau), A \subseteq f^{-1}(U)\}) \\
&\quad = \cap\{f^{-1}(U) \in I^X : U \in I^Y, f^{-1}(U) \in W^*_\alpha(\tau), A \subseteq f^{-1}(U)\} \\
&\quad \supseteq \cap\{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\} \\
&\quad = wcl_\alpha(A).
\end{aligned}$$

Hence $f(wcl_\alpha(A)) \subseteq wcl_\alpha(f(A))$.

(b) For every $A \in I^Y$, we have

$$\begin{aligned}
&f^{-1}(wcl_\alpha(A)) \\
&\quad = f^{-1}(\cap\{U \in I^Y : U \in W^*_\alpha(\sigma), A \subseteq U\}) \\
&\quad \supseteq f^{-1}(\cap\{U \in I^Y : f^{-1}(U) \in W^*_\alpha(\tau), f^{-1}(A) \subseteq f^{-1}(U)\}) \\
&\quad = \cap\{f^{-1}(U) \in I^X : U \in I^Y, f^{-1}(U) \in W^*_\alpha(\tau), \\
&\qquad f^{-1}(A) \subseteq f^{-1}(U)\} \\
&\quad \supseteq \cap\{K \in I^X : K \in W^*_\alpha(\tau), f^{-1}(A) \subseteq K\} \\
&\quad = wcl_\alpha(f^{-1}(A)).
\end{aligned}$$

(c) For every $A \in I^Y$, we have

$$\begin{aligned}
&f^{-1}(wint_\alpha(A)) \\
&\quad = f^{-1}(\cup\{U \in I^Y : U \in W_\alpha(\sigma), U \subseteq A\}) \\
&\quad \subseteq f^{-1}(\cup\{U \in I^Y : f^{-1}(U) \in W_\alpha(\tau), f^{-1}(U) \subseteq f^{-1}(A)\}) \\
&\quad = \cup\{f^{-1}(U) \in I^X : U \in I^Y, f^{-1}(U) \in W_\alpha(\tau), \\
&\qquad f^{-1}(U) \subseteq f^{-1}(A)\} \\
&\quad \subseteq \cup\{K \in I^X : K \in W_\alpha(\tau), K \subseteq f^{-1}(A)\} \\
&\quad = wint_\alpha(f^{-1}(A)).
\end{aligned}$$

□

4. Types of weak* smooth compactness

In this section, we introduce the concepts of several types of weak* smooth compactness in smooth topological spaces and investigate some of their properties.

We define the families $W_{w\alpha}(\tau) = \{A \in I^X : A = wint_\alpha(A)\}$ and $W^*_{w\alpha}(\tau) = \{A \in I^X : A = wcl_\alpha(A)\}$, where (X, τ) is a s.t.s. and $\alpha \in [0, 1)$. Then

$A \in W_{w\alpha}(\tau) \Leftrightarrow A^c \in W^*_{w\alpha}(\tau)$,

$A \in W_\alpha(\tau) \Rightarrow A \in W(\tau) \Rightarrow A \in W_{w\alpha}(\tau)$,

$A \in W^*_\alpha(\tau) \Rightarrow A \in W^*(\tau) \Rightarrow A \in W^*_{w\alpha}(\tau)$.

Definition 4.1. Let (X,τ) and (Y,σ) be two smooth topological spaces and let $\alpha \in [0,1)$. A function $f : X \to Y$ is called weak* smooth α-continuous with respect to τ and σ iff $A \in W_{w\alpha}(\sigma) \Rightarrow f^{-1}(A) \in W_{w\alpha}(\tau)$ for every $A \in I^Y$.

Let (X,τ) and (Y,σ) be two smooth topological spaces. A function $f : X \to Y$ is weak* smooth α-continuous with respect to τ and σ iff $A \in W^*_{w\alpha}(\sigma) \Rightarrow f^{-1}(A) \in W^*_{w\alpha}(\tau)$ for every $A \in I^Y$.

Definition 4.2. Let (X,τ) and (Y,σ) be two smooth topological spaces and let $\alpha \in [0,1)$. A function $f : X \to Y$ is called weak* smooth α-open (resp., weak* smooth α-closed) with respect to τ and σ iff $A \in W_{w\alpha}(\tau) \Rightarrow f(A) \in W_{w\alpha}(\sigma)$ (resp., $A \in W^*_{w\alpha}(\tau) \Rightarrow f(A) \in W^*_{w\alpha}(\sigma)$) for every $A \in I^X$.

Definition 4.3. Let $\alpha \in [0,1)$. A s.t.s. (X,τ) is called weak* smooth compact iff every family in $W_{w\alpha}(\tau)$ covering X has a finite subcover.

Definition 4.4. Let $\alpha \in [0,1)$. A s.t.s. (X,τ) is called weak* smooth nearly compact iff for every family $\{A_i : i \in J\}$ in $W_{w\alpha}(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}(\overline{A_i})^o = 1_X$.

Definition 4.5. Let $\alpha \in [0,1)$. A s.t.s. (X,τ) is called weak* smooth almost compact iff for every family $\{A_i : i \in J\}$ in $W_{w\alpha}(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}\overline{A_i} = 1_X$.

Theorem 4.6. *Let (X,τ) and (Y,σ) be two smooth topological spaces, $\alpha \in [0,1)$ and $f : X \to Y$ a surjective and weak* smooth α-continuous function with respect to τ and σ. If (X,τ) is weak* smooth compact, then so is (Y,σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $W_{w\alpha}(\sigma)$ covering Y, i.e., $\cup_{i\in J}A_i = 1_Y$. Then $\cup_{i\in J}f^{-1}(A_i) = f^{-1}(1_Y) = 1_X$. Since $f : X \to Y$ is weak* smooth α-continuous with respect to τ and σ, $\{f^{-1}(A_i) : i \in J\} \subseteq W_{w\alpha}(\tau)$. Since (X,τ) is weak* smooth compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}f^{-1}(A_i) = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0}f^{-1}(A_i)) = \cup_{i\in J_0}f(f^{-1}(A_i)) = \cup_{i\in J_0}A_i$. Therefore (Y,σ) is weak* smooth compact. □

THEOREM 4.7. *Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. If a function $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, then $f : X \to Y$ is weak* smooth α-continuous with respect to τ and σ.*

Proof. Let $f : X \to Y$ be a weak smooth α-continuous function with respect to τ and σ. Then by Theorem 3.7 $f^{-1}(wint_\alpha(A)) \subseteq wint_\alpha(f^{-1}(A))$ for every $A \in I^Y$. Let $A \in W_{w\alpha}(\sigma)$, i.e., $A = wint_\alpha(A)$. Then $f^{-1}(A) = f^{-1}(wint_\alpha(A)) \subseteq wint_\alpha(f^{-1}(A))$. From the definition of weak smooth α-interior we have $wint_\alpha(f^{-1}(A)) \subseteq f^{-1}(A)$. Hence $f^{-1}(A) = wint_\alpha(f^{-1}(A))$, i.e., $f^{-1}(A) \in W_{w\alpha}(\tau)$. Therefore $f : X \to Y$ is weak* smooth α-continuous with respect to τ and σ. □

We obtain the following corollary from Theorem 4.6 and 4.7.

COROLLARY 4.8. *Let (X, τ) and (Y, σ) be two smooth topological spaces, $\alpha \in [0, 1)$ and $f : X \to Y$ a surjective and weak smooth α-continuous function with respect to τ and σ. If (X, τ) is weak* smooth compact, then so is (Y, σ).*

THEOREM 4.9. *Let $\alpha \in [0, 1)$. Then a weak* smooth nearly compact s.t.s. (X, τ) is weak* smooth almost compact.*

Proof. Let $\{A_i : i \in J\}$ be a family in $W_{w\alpha}(\tau)$ covering X. Since (X, τ) is weak* smooth nearly compact, there exists a finite subset J_0 of J such that $\cup_{i \in J_0} (\overline{A_i})^o = 1_X$. Since $(\overline{A_i})^o \subseteq \overline{A_i}$ for each $i \in J$ by Proposition 3.2[3], $1_X = \cup_{i \in J_0} (\overline{A_i})^o \subseteq \cup_{i \in J_0} \overline{A_i}$. So $\cup_{i \in J_0} \overline{A_i} = 1_X$. Hence (X, τ) is weak* smooth almost compact. □

THEOREM 4.10. *Let (X, τ) and (Y, σ) be two smooth topological spaces, $\alpha \in [0, 1)$ and $f : X \to Y$ a surjective and quasi-smooth continuous function with respect to τ and σ. If (X, τ) is weak* smooth almost compact, then so is (Y, σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $W_{w\alpha}(\sigma)$ covering Y, i.e., $\cup_{i \in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i \in J} f^{-1}(A_i)$. Since f is quasi-smooth continuous with respect to τ and σ, f is weak* smooth continuous with respect to τ and σ by Theorem 3.5 and 4.7. Hence $f^{-1}(A_i) \in W_{w\alpha}(\tau)$ for each $i \in J$. Since (X, τ) is weak* smooth almost compact, there exists a finite subset J_0 of J such that $\cup_{i \in J_0} \overline{f^{-1}(A_i)} = 1_X$.

From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0}\overline{f^{-1}(A_i)}) = \cup_{i\in J_0}f(\overline{f^{-1}(A_i)})$. Since $f : X \to Y$ is quasi-smooth continuous with respect to τ and σ, from Proposition 3.5[3] we have $\overline{f^{-1}(A_i)} \subseteq f^{-1}(\overline{A_i})$ for each $i \in J$. Hence $1_Y = \cup_{i\in J_0}f(\overline{f^{-1}(A_i)}) \subseteq \cup_{i\in J_0}f(f^{-1}(\overline{A_i})) = \cup_{i\in J_0}\overline{A_i}$, i.e., $\cup_{i\in J_0}\overline{A_i} = 1_Y$. Thus (Y, σ) is weak* smooth almost compact. □

THEOREM 4.11. *Let (X, τ) and (Y, σ) be two smooth topological spaces, $\alpha \in [0, 1)$ and $f : X \to Y$ a surjective, quasi-smooth continuous and smooth open function with respect to τ and σ. If (X, τ) is weak* smooth nearly compact, then so is (Y, σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $W_{w\alpha}(\sigma)$ covering Y, i.e., $\cup_{i\in J}A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i\in J}f^{-1}(A_i)$. Since f is quasi-smooth continuous with respect to τ and σ, f is weak* smooth continuous with respect to τ and σ by Theorem 3.5 and 4.7. Hence $f^{-1}(A_i) \in W_{w\alpha}(\tau)$ for each $i \in J$. Since (X, τ) is weak* smooth nearly compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}(\overline{f^{-1}(A_i)})^o = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0}(\overline{f^{-1}(A_i)})^o) = \cup_{i\in J_0}f((\overline{f^{-1}(A_i)})^o)$. Since $f : X \to Y$ is smooth open with respect to τ and σ, from Proposition 3.6[3] we have $f((\overline{f^{-1}(A_i)})^o) \subseteq (f(\overline{f^{-1}(A_i)}))^o$ for each $i \in J$. Since $f : X \to Y$ is quasi-smooth continuous with respect to τ and σ, from Proposition 3.5[3] we have $\overline{f^{-1}(A_i)} \subseteq f^{-1}(\overline{A_i})$ for each $i \in J$. Hence $1_Y = \cup_{i\in J_0}f((\overline{f^{-1}(A_i)})^o) \subseteq \cup_{i\in J_0}(f(\overline{f^{-1}(A_i)}))^o \subseteq \cup_{i\in J_0}(f(f^{-1}(\overline{A_i})))^o = \cup_{i\in J_0}(\overline{A_i})^o$, i.e., $\cup_{i\in J_0}(\overline{A_i})^o = 1_Y$. Thus (Y, σ) is weak* smooth nearly compact. □

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