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SOME PROPERTIES OF WEIGHTED HARMONIC BERGMAN FUNCTIONS ON HALF-SPACES

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Abstract. On the setting of the upper half-space of the euclidean space $\mathbf{R}^n$, we show some properties of weighted harmonic Bergman functions.

1. Introduction

For a fixed positive integer $n > 1$, let $\mathbf{H} = \mathbf{R}^{n-1} \times \mathbf{R}_+$ be the upper half-space where $\mathbf{R}_+$ denotes the set of all positive real numbers. We write point $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbf{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$ and $1 \leq p < \infty$, let $b^p_\alpha(\mathbf{H})$ denote *weighted harmonic Bergman space* consisting of all real-valued harmonic functions u on $\mathbf{H}$ such that

$$||u||_{L^p_\alpha} := \left(\int_{\mathbf{H}} |u(z)|^p \, dV_\alpha(z) \right)^{1/p} < \infty,$$

where $dV_\alpha(z) = z_n^\alpha \, dz$ and dz is the Lebesgue measure on $\mathbf{R}^n$. We let $b^p_\alpha = b^p_\alpha(\mathbf{H})$. Then we can check easily that the space b^p_α is a Banach space with the usual weighted L^p-norm.

In this paper, we show some properties of b^p_α as stated below. In section 2, we review some basic results of the extended Poisson kernel. In section 3 we show the b^1_α-cancellation property, i.e., If $u \in b^1_\alpha$, then $\int_{\mathbf{H}} u(z) \, dV_\alpha(z) = 0$. we also find a necessary and sufficient condition for

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the space b^p_α to have a positive harmonic function and then we show that b^p_α is not contained in b^q_α if q is different from p.

Constants. Throughout the paper we use the same letter C to denote various constants which may change at each occurrence. The constant C may often depend on the dimension n and some other parameters, but it is always independent of particular points or parameters under consideration. For nonnegative quantities A and B, we often write $A \lesssim B$ or $B \gtrsim A$ if A is dominated by B times some *inessential* positive constant. Also, we write $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

2. Preliminary Results

Let $P(z,w)$ be the extended Poisson kernel on $\mathbf{H}$, i.e.,

$$P_z(w) := P(z,w) = \frac{2}{nV(B)} \frac{z_n + w_n}{|z-\overline{w}|^n} \tag{2.1}$$

where $V(B)$ is the volume of the unit ball in $\mathbf{R}^n$, $z \in \mathbf{H}$, $w \in \overline{\mathbf{H}} = \mathbf{H} \cup \partial\mathbf{H}$, and $\overline{w} = (w', -w_n)$. Here $\partial\mathbf{H} = \mathbf{R}^{n-1}$ denote the boundary of $\mathbf{H}$. Note that for each fixed $w \in \overline{\mathbf{H}}$, $P(z,w)$ is a positive and harmonic function on $\mathbf{H}$ as a function of z. Note also that for each $z \in \mathbf{H}$ and for every $w \in \overline{\mathbf{H}}$,

$$\int_{\partial\mathbf{H}} P(z,w)\, dw' = 1. \tag{2.2}$$

Also, we can show from (**??**) that for nonnegative integer k,

$$D_n^k P(z,0) = \frac{f_k(z)}{|z|^{n+2k}}, \tag{2.3}$$

where f_k is a homogeneous polynomial of degree $1+k$.

The poisson integral of $f \in L^p(\partial\mathbf{H})$, for $1 \le p \le \infty$, is the function $P[f]$ on $\mathbf{H}$ defined by

$$P[f](z) = \int_{\partial\mathbf{H}} P(z,t) f(t)\, dt.$$

Let k be a nonnegative integer. If $u \in b^p_\alpha$, then we know from the mean value property, Jensen's inequality and then Cauchy's estimate that

$$|D_n^k u(z)| \lesssim z_n^{-(n+\alpha)/p-k}$$

for each $z \in \mathbf{H}$. This shows that if $u \in b^p_\alpha$, then u is a bounded harmonic function on every proper half-space contained in $\mathbf{H}$. Thus we have

$$P[u(\cdot, z_n)](z', \delta) = u(z', z_n + \delta) \tag{2.4}$$

for $\delta > 0$. (See [?] and [?] for details and related facts.)

3. Some Properties of b^p_α

In this section, we show some properties of the weighted harmonic Bergman functions. First we show the b^1_α-cancellation property. To do so, we need a lemma.

LEMMA 1. *If $u \in b^1_\alpha$, then $\widetilde{u}$ is decreasing on $(0, \infty)$, where*

$$\widetilde{u}(\delta) = \int_{\partial\mathbf{H}} \big|u(t, \delta)\big|\, dt$$

for $\delta > 0$.

Proof. Suppose $0 < \delta_1 < \delta_2$. Then we know from (**??**) that

$$u(t, \delta_2) = P[u(\cdot, \delta_1)](t, \delta_2 - \delta_1).$$

Therefore we have

$$\big|u(t, \delta_2)\big| \le \int_{\partial\mathbf{H}} \big|u(s, \delta_1)\big| P\big((t, \delta_2 - \delta_1), s\big)\, ds.$$

Note that

$$P\big((t, \delta_2 - \delta_1), s\big) = P\big((s, \delta_2 - \delta_1), t\big)$$

for every $s, t \in \partial\mathbf{H}$. Integrating with respect to t and then using Fubini's theorem, we see from (**??**) that

$$\begin{aligned}
\widetilde{u}(\delta_2) &= \int_{\partial H} \big|u(t, \delta_2)\big|\, dt \\
&\le \int_{\partial H} \big|u(s, \delta_1)\big| \int_{\partial H} P\big((s, \delta_2 - \delta_1), t\big)\, dt\, ds \\
&= \widetilde{u}(\delta_1),
\end{aligned}$$

as desired. Therefore the proof is complete. □

As the above proof shows, the result of Lemma **??** holds if we only assume that u equals the Poisson integral of its boundary values on every proper half-space contained in $\mathbf{H}$.

Now we are ready to prove the following theorem.

THEOREM 2. *If $u \in b^1_\alpha$, then*

$$\int_{\partial \mathbf{H}} u(t,\delta)\,dt = 0$$

for each $\delta > 0$.

Proof. Fix $\delta > 0$. Then we know from Lemma **??** that $u(\cdot,\delta) \in L^1(\partial\mathbf{H})$. Also, we know from (**??**) that

$$u(z', z_n + \delta) = P[u(\cdot,\delta)](z)$$

for every $z \in \mathbf{H}$. Therefore we have from Fubini's theorem and (**??**) that

$$\begin{aligned}
\int_{\mathbf{H}} u(z', z_n + \delta)\,dz &= \int_0^\infty \int_{\partial\mathbf{H}} \int_{\partial\mathbf{H}} P(z,t) u(t,\delta)\,dt\,dz'\,dz_n \\
&= \int_0^\infty \int_{\partial\mathbf{H}} \left(\int_{\partial\mathbf{H}} P\big((t,z_n),z'\big)\,dz' \right) u(t,\delta)\,dt\,dz_n \\
&= \int_0^\infty \int_{\partial\mathbf{H}} u(t,\delta)\,dt\,dz_n. \qquad (3.1)
\end{aligned}$$

Because the inner integral in (**??**) is independent of z_n, we must have

$$\int_{\partial\mathbf{H}} u(t,\delta)\,dt = 0.$$

Therefore the proof is complete. □

As a corollary to the above theorem, we easily get the following b^1_α-cancellation property.

COROLLARY 3. *If $u \in b^1_\alpha$, then*

$$\int_{\mathbf{H}} u(z)\,dV_\alpha(z) = 0.$$

Now we find a necessary and sufficient condition for the space b^p_α to have a positive harmonic function. This means that certain b^p_α spaces do not contain any positive functions on the upper half-space and we can not have this property on bounded domains.

THEOREM 4. *b^p_α contains a positive harmonic function if and only if $p > (n+\alpha)/(n-1)$.*

Proof. Suppose that $p > (n+\alpha)/(n-1)$. Let $z_0 = (0,1)$ and let $u(z) = P(z, z_0)$ for $z \in \mathbf{H}$. Then clearly, u is a positive harmonic function on $\mathbf{H}$. Note from (**??**) that

$$\left|P(z,z_0)\right|^{p-1} \lesssim (z_n+1)^{-(n-1)(p-1)}.$$

Therefore we have from (**??**) that

$$\begin{aligned}
\|u\|^p_{L^p_\alpha} &= \int_{\mathbf{H}} |u(z)|^p z_n^\alpha \, dz \\
&\lesssim \int_0^\infty \int_{\partial\mathbf{H}} P(z,z_0)\, dz' \frac{z_n^\alpha}{(z_n+1)^{(n-1)(p-1)}}\, dz_n \\
&= \int_0^\infty \frac{z_n^\alpha}{(z_n+1)^{(n-1)(p-1)}}\, dz_n. \qquad (3.2)
\end{aligned}$$

Because $\alpha > -1$ and $(n-1)(p-1)-\alpha > 1$, the integral in (**??**) is finite. Hence we see that $u \in b^p_\alpha$ as desired.

Conversely, suppose $u \in b^p_\alpha$ is positive on $\mathbf{H}$. Then we know from [**?**] that

$$u(z) = cz_n + P[\mu](z)$$

for all $z \in \mathbf{H}$, where c is a nonnegative constant and μ is a positive Borel measure on $\partial\mathbf{H}$ satisfying

$$\int_{\partial\mathbf{H}} \frac{d\mu(t)}{(1+|t|)^n} < \infty.$$

Because $u \in b^p_\alpha$, we must have $c = 0$. Since u is a positive harmonic function, μ can not be the zero measure and so we can choose a compact set $K \subset \partial\mathbf{H}$ satisfying $\mu(K) > 0$. Let $R = \max\{\, |t| : t \in K\}$. Then we see that

$$\begin{aligned}
u(z) &\geq \frac{2z_n}{nV(B)\big(|z|+R\big)^n}\mu(K) \\
&\gtrsim \frac{z_n}{(|z|+1)^n}
\end{aligned}$$

on $\mathbf{H}$. Thus we have from (**??**) that

$$\begin{aligned}\infty &> \int_{\mathbf{H}} |u(z)|^p z_n^\alpha \, dz \\ &\gtrsim \int_{\mathbf{H}} \frac{z_n^{p+\alpha}}{(|z|+1)^{np}} \, dz \\ &\gtrsim \int_0^\infty \frac{z_n^{p+\alpha}}{(z_n+1)^{np-n+1}} \, dz.\end{aligned}$$

Because $u \in b^p$, we must have $np-n+1-p-\alpha > 1$, i.e., $p > n/(n-1)$. This completes the proof. □

We close this paper by showing that on $\mathbf{H}$, no Bergman space is properly contained in another.

THEOREM 5. *If $p \neq q$, then b^p_α does not contain b^q_α.*

Proof. Suppose to the contrary that $b^p_\alpha \subset b^q_\alpha$. Because convergence in any Bergman space implies uniform convergence on compact subsets, the closed graph theorem shows that the identity map from b^p_α to b^q_α is bounded. Thus there exists a positive constant C satisfying

$$\|v\|_{L^q_\alpha} \le C\|v\|_{L^p_\alpha} \tag{3.3}$$

for all $v \in b^p_\alpha$.

To show that (**??**) fails, we choose a nonnegative integer k large enough so that

$$(n+k-1)p > n+\alpha, \qquad (n+k-1)q > n+\alpha. \tag{3.4}$$

Set $u(z) = D_n^k P(z,0)$ for $z \in \mathbf{H}$. Then clearly u is harmonic on $\mathbf{H}$ and we see from (**??**) that

$$u(z) = \frac{f_k(z)}{|z|^{n+2k}}$$

for some homogenous polynomial of degree $k+1$. Let $u_\delta(z) = u\big(z + (0,\delta)\big)$ for $\delta > 0$. Then clearly u_δ is harmonic on $\mathbf{H}$. We also see from the homogeneity of f that

$$\begin{aligned}\|u_\delta\|^p_{L^p_\alpha} &= \int_{\mathbf{H}} \frac{\big|f\big(z+(0,\delta)\big)\big|^p}{\big|z+(0,\delta)\big|^{(n+2k)p}} z_n^\alpha \, dz \\ &= \frac{\delta^{n+(k+1)p+\alpha}}{\delta^{(n+2k)p}} \int_{\mathbf{H}} \frac{\big|f\big(z+(0,1)\big)\big|^p}{\big|z+(0,1)\big|^{(n+2k)p}} z_n^\alpha \, dz. \end{aligned} \tag{3.5}$$

We see from (**??**) that the integral in (**??**) is finite. Thus,

$$\|u_\delta\|_{L^p_\alpha} \approx \delta^{(n+\alpha)/p-n-k+1}$$

and similarly we have

$$\|u_\delta\|_{L^q_\alpha} \approx \delta^{(n+\alpha)/q-n-k+1}.$$

Therefore

$$\frac{\|u_\delta u\|_{L^q_\alpha}}{\|u_\delta u\|_{L^p_\alpha}} \approx \delta^{(n+\alpha)(1/q-1/p)} \tag{3.6}$$

for all $\delta > 0$. Because $p \neq q$, the right side of (**??**) is not a bounded function of δ. Thus (**??**) fails and the proof is complete. □

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