Kangweon-Kyungki Math. Jour. **11** (2003), No. 2, pp. 79–84

UNIVALENT FUNCTIONS ON $\Delta = \{z : |z| > 1\}$

Sook Heui Jun

Abstract. In this paper, we obtain the sharp estimates for coefficients of harmonic, orientation-preserving, univalent mappings defined on $\Delta = \{z : |z| > 1\}$ when harmonic mappings are of bounded variation on $|z| = 1$.

1. Introduction

A continuous function $f = u + iv$ defined in a domain $D \subseteq \mathbb{C}$ is harmonic if u and v are real harmonic in D. Study of univalent harmonic functions is pioneered by J. G. Clunie and T. Sheil-Small. They[2] obtained a number of sharp results when a univalent harmonic orientation-preserving mapping f defined in $\mathbb{D} = \{z : |z| < 1\}$ is convex, convex in one direction, or close-to-convex. Hengartner and Schober[4] studied the class Σ of all complex-valued, harmonic, orientation-preserving, univalent mappings f defined on $\Delta = \{z : |z| > 1\}$, which are normalized at infinity by $f(\infty) = \infty$. Such functions admit the representation

$$f(z) = h(z) + \overline{g(z)} + Alog|z| \tag{1}$$

where $h(z) = \alpha z + \sum_{k=0}^{\infty} a_k z^{-k}$ and $g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k}$ are analytic in Δ and $0 \leq |\beta| < |\alpha|$, $|A|/2 \leq |\alpha| + |\beta|$. In addition, $a = \overline{f_{\overline{z}}}/f_z$ is analytic and satisfies $|a(z)| < 1$. In this paper, we obtain the sharp bounds for the Fourier coefficients of (1) when the harmonic mapping $f \in \Sigma$ is of bounded variation on $|z| = 1$.

Received April 14, 2003.
2000 Mathematics Subject Classification: 30C15, 30C50.
Key words and phrases: Harmonic, univalent functions.
This work was supported by the research fund of Nat. Sci. Inst., Seoul Women's University, 2001.

2. Some Coefficient Estimates

Every homeomorphism of the unit circle onto a convex Jordan curve extends continuously to a univalent harmonic mapping of the unit disk $\mathbb{D}$ onto a convex domain bounded by a Jordan curve. T. Radó posed this as a problem. H. Kneser[5] gave an elegant solution which uses the monodromy theorem to deduce global univalence from local univalence. In the converse direction, a univalent harmonic function which maps the unit disk $\mathbb{D}$ onto a strictly convex domain bounded by a Jordan curve has a continuous extension to $\overline{\mathbb{D}}$[1]. We also have the same result for the function $f \in \Sigma$ which maps onto the exterior U of a strictly convex Jordan curve Γ.

THEOREM 2.1. *Suppose that f is a complex-valued, harmonic, orientation-preserving, univalent mapping from $\Delta = \{z : |z| > 1\}$ onto the exterior U of a strictly convex Jordan curve Γ with $f(\infty) = \infty$. Then f has a continuous extension to $\overline{\Delta}$.*

Proof. Since $f \in \Sigma$, f has the representation (1). Consider $f(z) - \alpha z - \overline{\beta z} - Alog|z|$. Then $f(1/z) - \alpha/z - \overline{\beta/z} + Alog|z|$ is a bounded harmonic function in $|z| < 1$. Thus $Lim_{r\to 1^-}\{f(r^{-1}e^{-i\theta}) - \alpha r^{-1}e^{-i\theta} - \overline{\beta}r^{-1}e^{i\theta} + Alogr\}$ exists a.e. Therefore the radial limits $Lim_{r\to 1^+} f(re^{i\theta})$ exists a.e. and belongs to Γ. A univalent analytic mapping ϕ from Δ onto U extends homeomorphically to $\overline{\Delta}$, and $\phi(\partial\Delta) = \Gamma$. Let $\Phi = \phi^{-1}$ in U. Then $\Phi\circ f$ is an orientation-preserving homeomorphism of Δ onto itself. Its radial limit function k exists and has modulus 1 a.e. on $\partial\Delta$. By redefining k on a set of measure zero, we may write $k(e^{i\theta}) = e^{i\eta(\theta)}$, where η is a nondecreasing function on $\mathbb{R}$ and $\eta(\theta + 2\pi) = \eta(\theta) + 2\pi$. Define E to be the at most countable set of points $e^{i\theta}$ on $\partial\Delta$ that correspond to the discontinuities, which are finite jumps, of η. On $(\partial\Delta) \setminus E$ the function $\phi \circ k$ is continuous and its values belong to Γ. At the points of the countable set E, the function $\phi \circ k$ has one-sided limits, which also belong to Γ since Γ is closed. Now $\phi \circ k$ and the radial limit function of f agree almost everywhere, and so

$$f(z) - \alpha z - \overline{\beta z} - Alog|z| = -\frac{1}{2\pi}\int_0^{2\pi} Re\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right](\phi\circ k(e^{i\theta}) - \alpha e^{i\theta} - \overline{\beta}e^{-i\theta})d\theta.$$

Thus the unrestricted limits

$$Lim_{z\to e^{i\theta}}\{f(z) - \alpha z - \overline{\beta z} - Alog|z|\}$$

exist and are equal to $\phi\circ k-\alpha e^{i\theta}-\overline{\beta}e^{-i\theta}$ at all points of $(\partial\Delta)\setminus E$. Therefore we conclude that the unrestricted limits $\hat{f}(e^{i\theta}) \equiv Lim_{z\to e^{i\theta}} f(z)$ exist and are equal to $\phi \circ k$ at all points of $(\partial\Delta) \setminus E$. Next, let $e^{i\theta_0}$ belong to E. Then the cluster set of f at θ_0 is the straight-line segment joining $A_0 = Lim_{\theta\uparrow\theta_0}\hat{f}(e^{i\theta})$ to $B_0 = Lim_{\theta\downarrow\theta_0}\hat{f}(e^{i\theta})$. If $A_0 = B_0$, then the cluster set is a singleton; so f has a limit and $\hat{f}$ is continuous there. If $A_0 \neq B_0$, then Γ would have to contain line segments corresponding to points of the discontinuity set. Since U^c is assumed to be strictly convex, the discontinuity set must be empty. Therefore f extends continuously to $\overline{\Delta}$. □

Lemma 2.2. *If $f \in \Sigma$ and f extends to be of bounded variation on $|z| = 1$, then $L_r \leq L_1 + 6\pi(|\alpha| + |\beta|)$ for $1 \leq r \leq 2$ where L_r denotes the length of $f(|z| = r)$.*

Proof. If $\psi(z) = f(z) - \alpha z - \overline{\beta z} - Alog|z|$, then $\psi(\infty) = a_0$. For any partition $P = [t_0, t_1, \ldots, t_N]$ of $[0, 2\pi]$, the expression $\sum_{k=1}^{N} |\psi(ze^{it_k}) - \psi(ze^{it_{k-1}})|$ is a subharmonic function of z in $\Delta \cup \{\infty\}$. Hence, by the Maximum Principle for subharmonic functions, we have

$$\sum_{k=1}^{N} |\psi(ze^{it_k}) - \psi(ze^{it_{k-1}})| \leq \limsup_{|z|\to 1} \sum_{k=1}^{N} |\psi(ze^{it_k}) - \psi(ze^{it_{k-1}})|.$$

Since

$$\sum_{k=1}^{N} |\psi(ze^{it_k}) - \psi(ze^{it_{k-1}})|$$
$$= \sum_{k=1}^{N} |f(ze^{it_k}) - f(ze^{it_{k-1}}) - \alpha(ze^{it_k} - ze^{it_{k-1}}) - \overline{\beta(ze^{it_k} - ze^{it_{k-1}})}|$$

, we have

$$\begin{aligned}
&\sum_{k=1}^{N}\{|f(ze^{it_k})-f(ze^{it_{k-1}})|-|e^{it_k}-e^{it_{k-1}}|(|\alpha|+|\beta|)|z|\}\\
\leq&\sum_{k=1}^{N}|\psi(ze^{it_k})-\psi(ze^{it_{k-1}})|\\
\leq&\limsup_{|z|\to 1}\sum_{k=1}^{N}|\psi(ze^{it_k})-\psi(ze^{it_{k-1}})|\\
\leq&\limsup_{|z|\to 1}\sum_{k=1}^{N}\{|f(ze^{it_k})-f(ze^{it_{k-1}})|+(|\alpha|+|\beta|)|z||e^{it_k}-e^{it_{k-1}}|\}\\
\leq&L_1+2\pi(|\alpha|+|\beta|).
\end{aligned}$$

Thus this implies that

$$\sum_{k=1}^{N}|f(ze^{it_k})-f(ze^{it_{k-1}})|\leq L_1+2\pi(|\alpha|+|\beta|)(1+|z|).$$

Let $z=r$, then

$$\sum_{k=1}^{N}|f(re^{it_k})-f(re^{it_{k-1}})|\leq L_1+2\pi(|\alpha|+|\beta|)(1+r).$$

Since P is arbitrary, we have $L_r\leq L_1+2\pi(|\alpha|+|\beta|)(1+r)$. For $r\leq 2,\ L_r\leq L_1+6\pi(|\alpha|+|\beta|)$. □

THEOREM 2.3. *If $f\in\Sigma$ and f extends to be of bounded variation on $|z|=1$, then*

$$|\alpha+\overline{b}_1|\leq\frac{L_1}{2\pi},\qquad|\beta+\overline{a}_1|\leq\frac{L_1}{2\pi},$$

$$|b_n|\leq\frac{L_1}{2n\pi}\quad\text{and}\quad|a_n|\leq\frac{L_1}{2n\pi}\qquad\text{for}\ \ n\geq 2,$$

where L_1 is the length of $f(|z|=1)$. The first inequality $|\alpha+\overline{b}_1|\leq L_1/(2\pi)$ is sharp for the function $f(z)=z+i/(2\overline{z})+(1/2)log|z|$. The *inequality* $|\beta+\overline{a}_1| \le L_1/(2\pi)$ *is sharp for the function* $f(z) = z - 1/\overline{z} + 2log|z|$. *The inequalities* $|b_n| \le L_1/(2n\pi)$ *and* $|a_n| \le L_1/(2n\pi)$ *for* $n \ge 2$ *are sharp for the function* $f(z) = z - 1/\overline{z} + 2arg\left(\frac{1+i/z}{1-i/z}\right)$.

Proof. Let n be any nonzero integer. Lemma 2.2 implies that the L_r's are uniformly bounded for $1 \le r \le 2$. By the Helly selection theorem, there exists a sequence $\langle r_k \rangle$ such that $r_k \searrow 1$ and $\int\limits_{|z|=r_k} z^n df \to \int\limits_{|z|=1} z^n df$ as $k \to \infty$. Since

$$\begin{aligned}\int\limits_{|z|=r_k} z^n df &= \int\limits_{|z|=r_k} z^n (f_z dz + f_{\overline{z}} d\overline{z}) \\ &= \int\limits_{|z|=r_k} z^n \Bigg\{(\alpha + \frac{A}{2z} + \sum_{k=1}^{\infty} a_k(-k)z^{-k-1})dz \\ &\qquad + (\overline{\beta} + \frac{A}{2\overline{z}} + \sum_{k=1}^{\infty} \overline{b}_k(-k)\overline{z}^{-k-1})d\overline{z}\Bigg\} \\ &= \begin{cases} 2\pi i(\alpha + \overline{b}_1 r_k^{-2}) & \text{if } n = -1 \\ 2\pi i(-n\overline{b}_{-n} r_k^{2n}) & \text{if } n \le -2 \\ 2\pi i(-a_1 - r_k^2\overline{\beta}) & \text{if } n = 1 \\ 2\pi i(-na_n) & \text{if } n \ge 2, \end{cases}\end{aligned}$$

it follows that

$$\frac{1}{2\pi i}\int\limits_{|z|=1} z^n df = \begin{cases} \alpha + \overline{b}_1 & \text{if } n = -1 \\ -n\overline{b}_{-n} & \text{if } n \le -2 \\ -a_1 - \overline{\beta} & \text{if } n = 1 \\ -na_n & \text{if } n \ge 2. \end{cases}$$

Since $|\int\limits_{|z|=1} z^n df| \le \int\limits_{|z|=1} |df| = L_1$, we have

$$|\alpha + \overline{b}_1| \le \frac{L_1}{2\pi}, \qquad |\beta + \overline{a}_1| \le \frac{L_1}{2\pi},$$

$$n|b_n| \le \frac{L_1}{2\pi} \quad \text{and} \quad n|a_n| \le \frac{L_1}{2\pi} \quad \text{for } n \ge 2.$$

These inequalities are equivalent to the desired ones. □

Corollary 2.4. *If $f \in \Sigma$ and $\mathbb{C}\backslash f(\Delta)$ is strictly convex, then*

$$|\alpha + \bar{b}_1| \le \frac{L}{2\pi}, \qquad |\beta + \bar{a}_1| \le \frac{L}{2\pi},$$

$$n|b_n| \le \frac{L}{2\pi} \quad \text{and} \quad n|a_n| \le \frac{L}{2\pi} \qquad \text{for} \ \ n \ge 2,$$

where L is the length of $\partial(\mathbb{C}\backslash f(\Delta))$.

Proof. Since $f \in \Sigma$ and $\mathbb{C}\backslash f(\Delta)$ is strictly convex, f has a continuous extension to $\overline{\Delta}$ by Theorem 2.1. Thus f extends to be of bounded variation on $|z| = 1$. The equalities follow directly from Theorem 2.3.□

Corollary 2.5. *If $f \in \Sigma$ and $f(\Delta) = \Delta$, then*

$$|\alpha + \bar{b}_1| \le 1, \qquad |\beta + \bar{a}_1| \le 1,$$

$$|b_n| \le \frac{1}{n} \quad \text{and} \quad |a_n| \le \frac{1}{n} \qquad \text{for} \ \ n \ge 2.$$

Proof. Since $\mathbb{C}\backslash\Delta$ is strictly convex and the unit circle has the length 2π, the inequalities follow directly from Corollary 2.4. □

References

1. G. Choquet, *Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques*, Bull. Sci. Math.(2) **69** (1945), 156-165.
2. J. G. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A.I **9** (1984), 3-25.
3. P. Duren and G. Schober, *A variational method for harmonic mappings onto convex regions*, Complex Variables Theory Appl. **9** (1987), 153-168.
4. W. Hengartner and G. Schober, *Univalent harmonic functions*, Trans. Amer. Math. Soc. **299** (1987), 1-31.
5. H. Kneser, *Lösung der Aufgabe 41*, Jahresber. Deutsch. Math.-Verein. **35** (1926), 123-124.

Department of Mathematics
Seoul Women's University
126 Kongnung 2-dong, Nowon-Gu
Seoul, 139-774, Korea