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DYNAMICS OF COUNTING

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Abstract. In this paper we are going to study the dynamics of counting on the set $\mathcal{S}$ of functions from a finite subset of $\mathbb{N} = \{1, 2, \cdots\}$ into $\mathbb{N}$. We have shown that every point $f \in \mathcal{S}$ is either an eventually fixed point or an eventually periodic point of period 2 or 3.

1. Notation and Object

We denote by $[1, n]$ the set of integers $\{1, 2, \cdots, n\}$. We represent a function $f : [1, n] \to \mathbb{N}$ by

$$f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$$

If A is a finite subset of $\mathbb{N}$, we denote by $\#(A)$ the number of elements of the set A. Let $\mathcal{S}$ be the set of all functions $f : A \to \mathbb{N}$ from a finite nonempty subset of $\mathbb{N}$ into $\mathbb{N}$. We define a counting function $C : \mathcal{S} \to \mathcal{S}$ in the following way:

(1) $\operatorname{dom} C(f) = \operatorname{dom} f \cup \operatorname{im} f$.
(2) For each $k \in \operatorname{dom} C(f)$,

$$C(f)(k) = \begin{cases} 1, \text{ if } k \notin \operatorname{im} f, \\ \#\big(f^{-1}(k)\big), \text{ if } k \in \operatorname{im} f \setminus f^{-1}(k), \\ 1 + \#\big(f^{-1}(k)\big), \text{ if } k \in \operatorname{im} f \cap f^{-1}(k). \end{cases}$$

Question. Given $f \in \mathcal{S}$, let $f_0 = f,\ f_n = C(f_{n-1}), n \geq 1$. What is the property of the orbit of f

$$\{f_0, f_1, f_2, \cdots\}.$$

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2. Main Result

THEOREM 2.1. *For each $f : [1,n] \to [1,n]$, one of the following holds true:*

(1) $n \leq 4$: *f is an eventually fixed point.*

(2) $n = 5$:

(2-a) *If f is injective or constant, then f is an eventually periodic point of period 2.*

(2-b) *Otherwise f is an eventually fixed point.*

(3) $n = 6$:

(3-a) *If f is injective or constant, then f is an eventually periodic point of period 3.*

(3-b) *Otherwise f is an eventually fixed point of period 2.*

(4) $n = 7$:

(4-a) *If $\#\big(f^{-1}(1)\big) = n-4$ and $\#\big(f^{-1}(a)\big) = 2$ for some $a \neq 1$, then f is an eventually fixed point.*

(4-b) *If $\#\big(f^{-1}(1)\big) = 4, 5$, then f is an eventually fixed point.*

(4-c) *Otherwise f is an eventually periodic point of period 3.*

(5) $n \geq 8$:

(5-a) *If $\#\big(f^{-1}(1)\big) = n-4$ and $\#\big(f^{-1}(a)\big) = 2$ for some $a \neq 1$, then f is an eventually fixed point.*

(5-b) *Otherwise f is an eventually periodic point of period 2.*

LEMMA 2.1. *If $n \geq 7$, then for each $f : [1,n] \to [1,n]$ we have*

$$\#\big(f_2^{-1}(1)\big) \geq n-4$$

for some p.

Proof. If suffices to consider the case $\#\big(f^{-1}(1)\big) = n-k$, $4 < k < n$. Thus there are k points at which the values of f are different form 1. If these k values are all different, we have $\#\big(f_2^{-1}(1)\big) = n-3$. If f has the same value more than points of these k points, than $\#\big(f_2^{-1}(1)\big) = n-2$. If f has the same value at exactly two points of these k points, then $\#\big(f_2^{-1}(1)\big) \geq n-4$. □

LEMMA 2.2. *$f : [1,n] \to [1,n]$ and $\#\big(f^{-1}(1)\big) = n-1$.*

(1) *If $n = 5$, then f is an eventually fixed point.*

(2) *If $n \geq 6$, then f is an eventually periodic point of period 2.*

Proof. By hypotheses there exists exactly one a at which $f(a) \neq 1$. Therefore

$$f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ n-1 & 2 & 1 & \cdots & 1 & 2 & \end{pmatrix}.$$

If $n = 5$, we calculate to see that

$$f_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 3 & 1 & 1 \end{pmatrix} \text{ and } f_5 = f_4.$$

If $n \geq 6$, we have

$$f_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-3 & n-2 & n-1 & n \\ n-3 & 4 & 1 & 1 & 1 & \cdots & 1 & 2 & 1 & 1 \end{pmatrix},$$
$$f_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-3 & n-2 & n-1 & n \\ n-2 & 2 & 1 & 2 & 1 & \cdots & 2 & 1 & 1 & 1 \end{pmatrix},$$
$$f_7 = f_5.$$

This means that f is an eventually periodic point of period 2. □

COROLLARY 2.1. *If $f : [1, n] \to [1, n]$, $n \geq 5$, is injective or constant, then f is an eventually periodic point of period* 2.

Proof. If f is injective, then f is constant. Thus it suffices to consider the case that f is constant. But if f is constant, then $f_2 : [1, n+1] \to [1, n+1]$ and $\#(f_2^{-1}(1)) = n = n + 1 - 1$. Hence f is an eventually periodic point of period 2. □

Proof of Theorem 2.1. By Lemma 2.1, it suffices to consider the following cases:

(1) $\#(f^{-1}(1)) = n - 1$,
(2) $\#(f^{-1}(1)) = n - 2$,
(3) $\#(f^{-1}(1)) = n - 3$,
(4) $\#(f^{-1}(1)) = n - 4$

We have treated the case $\#(f^{-1}(1)) = n - 1$ in Lemma 2.2. The other cases by the same method. □

THEOREM 2.2. *For each $f \in \mathcal{S}$, f is either an eventually fixed point or an eventually periodic point of period 2 or 3.*

Proof. We first choose $n \geq 5$ so that dom $f \subset [1, n]$. Then

$$\bigcup_{k=1}^{\infty} \text{ im } f_k \subset [1, n+1].$$

By pigeon hole principle we have $f_p = f_q$ for some $p > q$. Therefore f is an eventually fixed point or an eventually periodic point. Observe that if $a, b \in$ dom f and there is no integer $k \in$ dom f such that $a < k < b$, then the orbit of $\tilde{f}$, $\tilde{f} : \big(\text{ dom } f \setminus \{b\}\big) \cup \{a+1\} \to \mathbb{N}$ and

$$\tilde{f}(k) = \begin{cases} f(k), \text{ if } k \neq a+1 \ , \\ f(b), \text{ if } k = a+1 \end{cases}$$

, has the same property of that of f. Therefore we may assume that dom $f = [1, n]$ for some $n \geq 4$. Now the conclusion follows from Theorem 2.1. □

3. Examples

EXAMPLE 3.1. Let $f : [1, n] \to [1 : n]$ is a fixed point. Then

$$\sum_{k=2}^{n} (k-1) f(k) = \frac{n(n+1)}{2}.$$

Proof. By the definition of $C(f)$ $f_(k)$ is the number of the k's in the upper and lower rows of the representations of f. Thus

$$\sum_{k=1}^{n} k f(k) = \sum_{k=1}^{n} \big(k + f(k)\big).$$

□

From Theorem 2.1 we observe that there are infinite number of eventually fixed point. Using the Lemma 3.1, we can find all the fixed points for $n \leq 10$.

EXAMPLE 3.2. The following are the only fixed points $f : [1, n] \to [1, n]$ when $n \le 10$.

Proof. If f is a fixed point, we have

$$\sum_{k=2}^{n}(k-1)x_k = \frac{n(n+1)}{2},$$

where $x_k = f(k)$. Thus, if we let $y_k = x_k - 1, k = 2, 3, \cdots, n$, we have

$$y_2 + 2y_3 + \cdots + (n-1)y_n = n.$$

This is a linear Diophantine equation and we find that

$$\begin{aligned} y_2 &= n - 2v_2 + v_3, \\ y_3 &= v_2 - 2v_3 + v_4, \\ \vdots &= \qquad \vdots \qquad , \\ y_{n-2} &= v_{n-3} - 2v_{n-2} + v_{n-1}, \\ y_{n-1} &= v_{n-2} - 2v_{n-1}, \\ y_n &= v_{n-1}, \end{aligned}$$

where $v_2, v_3, \cdots, v_{n-1}$ are any integers. However we are interested in the nonnegative integer solutions. A little more calculations show that:

(1) If $n \le 3$, then there is no fixed point.
(2) If $n = 4$, there are two fixed points

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 3 & 1 \end{pmatrix}.$$

(3) If $n = 5$, there is only one fixed point

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 3 & 1 & 1 \end{pmatrix}.$$

(4) If $n = 6$, there is no fixed point.

(5) if $n = 7$, there is only one fixed point.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 2 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

(6) If $n = 8$, there is only one fixed point.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 3 & 2 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

(7) If $n = 9$, there is only one fixed point.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 3 & 2 & 1 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

(8) If $n = 10$, there is only one fixed point.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 3 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

□

However $f : [1, n] \to [1, n]$ can be a periodic point of period 3 when $n = 6$ or 7.

EXAMPLE 3.3. The only periodic points of period 3 are the following:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 2 & 2 & 1 & 2 & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 2 & 2 & 1 & 2 & 1 \end{pmatrix}.$$

Proof. If $f : [1, n] \to [1, n]$ is a periodic point of period 3, we have $n = 6$ or $n = 7$ by Theorem 2.1. Furthermore

$$\sum_{k=1}^{n} k f_1(k) = \sum_{k=1}^{n} \big(k + f(k)\big),$$
$$\sum_{k=1}^{n} k f_2(k) = \sum_{k=1}^{n} \big(k + f_1(k)\big),$$
$$\sum_{k=1}^{n} k f(k) = \sum_{k=1}^{n} \big(k + f_2(k)\big).$$

Therefore

$$\sum_{k=2}^{n}(k-1)\big(f(k)+f_1(k)+f_2(k)\big)=\frac{3}{2}n(n+1).$$

By solving this Diophantine equations we obtain two points of period 3:

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 5 & 2 & 2 & 1 & 2 & 1 & 1\end{pmatrix}, \qquad \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 1 & 5 & 2 & 2 & 1 & 2 & 1\end{pmatrix}.$$

□

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