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DIAMETERS AND CLIQUE NUMBERS OF QUASI-RANDOM GRAPHS

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Abstract. We show that every quasi-random graph $G(n)$ with n vertices and minimum degree $(1 + o(1))n/2$ has diameter either 2 or 3 and that every quasi-random graph $G(n)$ with n vertices has a clique number of $o(n)$ with wide spread.

1. Introduction

Let us consider the random graph model for graphs with n vertices and edge probability $p = 1/2$. Thus the probability space $\Omega(n)$ consists of all labeled graphs G of order n, and the probability $P(G)$ of G in $\Omega(n)$ is given by $P(G) = 2^{-\binom{n}{2}}$. For a graph property $\mathcal{P}$, it may happen that

$$P\{G \in \Omega(n) \mid G \text{ satisfies } \mathcal{P}\} \to 1 \text{ as } n \to \infty.$$

In this case, a typical graph in $\Omega(n)$, which we denote by $G_{1/2}(n)$, has property $\mathcal{P}$ with overwhelming probability as n becomes large. We abbreviate this by saying that a random graph $G_{1/2}(n)$ *almost surely* has property $\mathcal{P}$. For details of these concepts, see [1] or [6].

One would like to construct graphs that behave just like a random graph $G_{1/2}(n)$. Of course, it is logically impossible to construct a truly random graph. Thus Chung, Graham, and Wilson defined in [3] quasi-random graphs, which simulate $G_{1/2}(n)$ without much deviation. Among many equivalent quasi-random properties studied in [3] and [2], we list only three needed in this paper. Let $G(n)$ denote a graph on n vertices. A family $\{G(n)\}$ of graphs (or for brevity, a graph $G = G(n)$) is said *quasi-random* if it satisfies any one of and hence all of the following.

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Uniform edge density property: For each subset $S \subseteq V(G)$, the number $e(S)$ of edges in $G[S]$ is $e(S) = \frac{1}{4}|S|^2 + o(n^2)$. Here, $G[S]$ denotes the subgraph of G induced by S.

Induced subgraph property: For fixed s, each labeled graph $M(s)$ on s vertices occurs $(1 + o(1))n^s/2^{\binom{s}{2}}$ times as an induced subgraph of G.

Uniform edge density property for bisectors: For each subset $S \subseteq V(G)$, the number $e(S, \overline{S})$ of edges between S and $\overline{S}$ satisfies $e(S, \overline{S}) = \frac{1}{2}|S||\overline{S}| + o(n^2)$, where $\overline{S} = V(G) - S$.

We showed in [4] and [5] how much quasi-random graphs deviate from random graphs $G_{1/2}(n)$ in connectedness and Hamiltonicity. In this paper, we investigate the same questions in diameter and clique number.

2. Diameters

In this section, we investigate the diameter of a quasi-random graph. It is well known that $G_{1/2}(n)$ has diameter 2 almost surely [1]. For quasi-random graphs $G(n)$, the diameter does not behave in quite the same way even with a degree restriction.

THEOREM 1. *Let $G = G(n)$ be a quasi-random graph and $\delta(G)$ denote the minimum degree of G. If $\delta(G) = (1 + o(1))n/2$, then the diameter of G is either* 2 *or* 3.

Proof. Let $\delta(G) = (1 + o(1))n/2$ and let u and v be vertices of G. Then for sufficiently large n, G is connected by Corollary 3 in [4] and so we may consider the diameter of G. If u and v have the neighborhoods $N(u)$ and $N(v)$ such that $N(u) \cap N(v) \neq \emptyset$, there is a u-v path of length at most 2. Now suppose that $N(u) \cap N(v) = \emptyset$. Then we have the disjoint union

$$V(G) = N(u) \cup N(v) \cup S,$$

where $S \subseteq V(G)$ with $|S| = o(n)$. Thus, using the uniform edge density property for bisectors, we have

$$\begin{aligned} e(N(u), N(v)) &= e(N(u), \overline{N(u)}) - e(N(u), S) \\ &\geq e(N(u), \overline{N(u)}) - |N(u)||S| \\ &= \frac{1}{2}|N(u)||\overline{N(u)}| + o(n^2) - |N(u)||S| \\ &\geq \frac{1}{2}(1 + o(1))\Big(\frac{n}{2}\Big)^2 + o(n^2) - \mathcal{O}(n)o(n) \\ &= (1 + o(1))\frac{n^2}{8}. \end{aligned}$$

Therefore, for sufficiently large n, there is an edge between $N(u)$ and $N(v)$ and hence we have a u-v path of length 3. Notice that the diameter of G cannot be 1. This implies that G has diameter 2 or 3 for sufficiently large n. □

We showed that every quasi-random graph $G(n)$ with minimum degree $(1 + o(1))n/2$ has diameter 2 or 3. Let p be a prime satisfying $p \equiv 1 \pmod 4$. We define Paley graph Q_p as follows: The vertex set is $\{0, 1, \ldots, p-1\}$ and ij is an edge precisely when $i - j$ is a quadratic residue of p. Note that Paley graph Q_p on p vertices is quasi-random [4] and strongly regular with parameters $\{(p-1)/2, (p-5)/4, (p-1)/4\}$ [1]. Here is an example of quasi-random graph whose diameter is 2.

Example 2. Any two non-adjacent vertices of Paley graph Q_p have $(p-1)/4$ common neighbors and any two adjacent vertices of Q_p have $(p-5)/4$ common neighbors. Hence Q_p has diameter 2 for all p.

The following is an example of quasi-random graph whose diameter is 3.

Example 3. Let us define $G(n)$ for $n = p + 2$ as follows. We add two new vertices u and v to Paley graph Q_p and next join u to all even numbered vertices of Q_p and v to all odd numbered vertices of Q_p. Then $G(n)$ is a quasi-random graph with $\delta = (1 + o(1))n/2$. Moreover, the distance between u and v is 3. Therefore, $G(n)$ has diameter 3 for all p.

The following example shows that there are quasi-random graphs with diameter larger than 3 if the degree condition in Theorem 1 is not considered.

Example 4. Let $k \geq 2$ be a fixed integer and let P_k be a path of order k that is disjoint from Q_p. We hitch an end vertex of P_k to a vertex of Q_p introducing a new edge. Then the resulting graph $G(n)$, $n = p+k$, is a connected quasi-random graph with $\delta = 1$ and has diameter $k+2$ for all p.

3. Clique Numbers

In this section, we investigate the clique number of a quasi-random graph.

A maximal complete subgraph of a graph is called a *clique* of the graph. The *clique number* $cl(G)$ of a graph G is the maximum order of a clique of G, equivalently, $cl(G)$ is simply the maximum order of a complete subgraph. The *independence number* $\beta(G)$ of a graph G is the maximum cardinality of an independent set of G. We know that a set S of vertices in a graph G is independent if and only if S is a clique in the complement $\overline{G}$. Thus, $\beta(G) = cl(\overline{G})$.

It is well known that for almost every random graph $G_{1/2}(n)$, the clique number is around $2\log_2 n$ (see [1]). For quasi-random graphs, we have the following from Theorem 1 in [5], which contrasts with the case of a random graph.

Theorem 5. *Let $G = G(n)$ be a quasi-random graph on n vertices. Then the clique number $cl(G)$ of G satisfies $cl(G) = o(n)$ and is bounded away from zero by any positive constant.*

Proof. To prove this, it suffices to show that the complement $\overline{G}(n)$ of a quasi-random graph $G(n)$ is quasi-random. To check the uniform edge density property, we let S be a set of vertices of $G(n)$. Then we have

$$\begin{aligned} e_{\overline{G}}(S) &= \binom{|S|}{2} - e_G(S) \\ &= \frac{|S|(|S|-1)}{2} - \Big(\frac{|S|^2}{4} + o(n^2)\Big) \\ &= \frac{|S|^2}{4} - \frac{|S|}{2} - o(n^2) \\ &= \frac{|S|^2}{4} + o(n^2), \end{aligned}$$

which means that $\overline{G}(n)$ is quasi-random. □

Example 6. Let Π be a projective geometry of dimension k over the field $\mathbb{F}_2$, that is, the set of all 1-dimensional subspaces of a $(k+1)$-dimensional vector space over $\mathbb{F}_2$. We define a graph G whose vertices are the points of Π and where two vertices are adjacent if and only if their scalar product is zero. This graph G may be regarded as follows: The vertex set of G is the set of all nonempty subsets of $\{1, 2, \ldots, k\}$ and two vertices x and y are adjacent if and only if $|x \cap y| \equiv 0 \pmod 2$.

Let H be the subgraph of G induced by subsets with odd number of elements. Then H is a quasi-random graph of order $n = 2^k$ [3]. Since a clique of H corresponds a set of linearly independent vectors, it is easy to see that $cl(H) \leq \log_2 n$.

Example 7. Let n be a square with $n \equiv 1 \pmod 4$. We define a graph G_n of order n as follows: The vertex set of G_n is the field $\mathbb{F}_n$ and two vertices x and y are adjacent if and only if $x - y$ is a nonzero square in $\mathbb{F}_n$. Then G_n is a quasi-random graph of order n [3]. Since the elements in the subfield of order $\sqrt{n}$ form a clique, it is easy to see that $cl(G_n) \geq \sqrt{n}$.

We know that the clique number $cl(G(n))$ of a quasi-random graph $G(n)$ of order n satisfies $cl(G(n)) = o(n)$ and that almost every random graph $G_{1/2}(n)$ has clique number around $2\log_2 n$. However, Examples 6 and 7 show that clique numbers of quasi-random graphs are widely spread. But it still remains to determine tight bounds for clique numbers of quasi-random graphs.

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