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TRANSLATION THEOREM ON FUNCTION SPACE

Jae Gil Choi and Young Seo Park

Abstract. In this paper, we use a generalized Brownian motion process to define a translation theorem. First we establish the translation theorem for function space integrals. We then obtain the general translation theorem for functionals on function space.

1. Introduction.

In [1], Cameron and Martin introduced the transformation of Wiener integrals under the translation. In [4], Chang, Skoug and Park studied translation theorems for Fourier-Feynman transforms and conditional Fourier-Feynman transforms.

In this paper, we study a translation theorem for functionals on function space but with x in a very general function space $C_{a,b}[0,T]$ rather than in the Wiener space. The Wiener process used in [1,4] is free of drift and is stationary in time while the stochastic processes used in this paper is nonstationary in time and is subject to a drift $a(t)$.

In Section 2 of this paper, we give the basic concepts and notations. In Section 3, we study a translation theorem for function space integrals. Finally, in Section 4, we establish the general translation theorem for functionals on function space.

2. Definitions and preliminaries.

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega)$=0 almost everywhere and

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for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega),\cdots,Y(t_n,\omega))$ is normally distributed with density function

$$\begin{aligned} K(\vec{t},\vec{\eta}) &= \Big((2\pi)^n \prod_{j=1}^{n}(b(t_j)-b(t_{j-1}))\Big)^{-1/2} \\ &\quad \cdot \exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}\frac{((\eta_j-a(t_j))-(\eta_{j-1}-a(t_{j-1})))^2}{b(t_j)-b(t_{j-1})}\Big\} \end{aligned} \tag{2.1}$$

where $\vec{\eta}=(\eta_1,\cdot\cdot\cdot,\eta_n)$, $\eta_0=0$, $\vec{t}=(t_1,\cdots,t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0)=0$, $a'(t)\in L^2[0,T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0)=0$ and $b'(t)>0$ for each $t\in[0,T]$.

As explained in [8, p.18-20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D,\mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real valued functions $x(t)$, $t\in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x)=x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D,\mathcal{B}^D,\mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t)=\min\{b(s),b(t)\}$. By Theorem 14.2 [8, p.187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0)=0$ under the sup norm). Hence $(C_{a,b}[0,T],\mathcal{B}(C_{a,b}[0,T]),\mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

Let $L^2_{a,b}[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$: i.e.,

$$L^2_{a,b}[0,T]=\Big\{v:\int_0^T v^2(s)db(s)<\infty \text{ and } \int_0^T v^2(s)d|a|(s)<\infty\Big\} \tag{2.2}$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$.

For convenience, let $BV[0,T]$ be the space of bounded variation functions on $[0,T]$. We denote the function space integral of a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional F by

$$\int_{C_{a,b}[0,T]} F(x)d\mu(x)$$

whenever the integral exists.

3. Translation theorem for function space integrals.

Let $(C_{a,b}[0,T],\ \mathcal{B}(C_{a,b}[0,T]),\mu)$ be the function space induced by the generalized Brownian motion process defined in Section 1. In this section we will obtain a translation theorem for function space integrals over $(C_{a,b}[0,T],\mathcal{B}(C_{a,b}[0,T]),\mu)$.

For a partition $\tau = \{t_1,\cdots,t_n\}$ of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$, define a function $X_\tau : C_{a,b}[0,T] \longrightarrow \mathbb{R}^n$ by $X_\tau(x) = (x(t_1),\cdots,x(t_n))$. For $x \in C_{a,b}[0,T]$, define the function $[X_\tau(x)] \equiv [x]_n : [0,T] \longrightarrow \mathbf{R}$ by

$$[x]_n(t) = x(t_{j-1}) + \frac{b(t)-b(t_{j-1})}{b(t_j)-b(t_{j-1})}(x(t_j)-x(t_{j-1})) \tag{3.1}$$

for each $t \in [t_{j-1},t_j], j = 1,2,\cdots,n$. In case, $[x]_n$ is called *the polygonalized form* of x. Similarly, for $\vec{\xi} = (\xi_1,\xi_2,\cdots,\xi_n) \in \mathbf{R}^n$, define the function $[\vec{\xi}]_n : [0,T] \longrightarrow \mathbf{R}$ by

$$[\vec{\xi}]_n(t) = \xi_{j-1} + \frac{b(t)-b(t_{j-1})}{b(t_j)-b(t_{j-1})}(\xi_j-\xi_{j-1}) \tag{3.2}$$

for each $t \in [t_{j-1},t_j], j = 1,2,\cdots,n$, and $\xi_0 = 0$.

For any positive integer n, define the point t_j by

$$t_j = \frac{T}{n}j$$

where $j = 1,2,\cdots,n$.

LEMMA 3.1. *Let $[x]_n$ be the polygonalized form of x as in (3.1) and let F be a bounded and continuous functional on $C_{a,b}[0,T]$. Then*

$$\lim_{n\to\infty}\int_{C_{a,b}[0,T]} F([x]_n)d\mu(x) = \int_{C_{a,b}[0,T]} F(x)d\mu(x). \tag{3.3}$$

Proof. Let $\varepsilon > 0$ be given. For any $x \in C_{a,b}[0,T]$, there exists an integer $n_0 = n_0(\varepsilon)$ such that for all $|t' - t''| \le 1/n_0$, we have

$$|x(t') - x(t'')| < \frac{\varepsilon}{2}. \tag{3.4}$$

By using (3.1) and (3.4), we have for each $n \ge n_0$ and $t_{j-1} \le t \le t_j$,

$$\begin{aligned} |x(t) - [x]_n(t)| &= \left| x(t) - x(t_{j-1}) - \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}(x(t_j) - x(t_{j-1}))\right| \\ &\le |x(t) - x(t_{j-1})| + \left|\frac{b(t_j) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}\right| |x(t_j) - x(t_{j-1})| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{3.5}$$

Hence

$$\lim_{n\to\infty}[x]_n(t) = x(t) \tag{3.6}$$

uniformly on $[0,T]$. Since F is continuous on $C_{a,b}[0,T]$,

$$\lim_{n\to\infty} F([x]_n) = F(x). \tag{3.7}$$

Let $F_n(x) = F([x]_n)$. Then for all $n \in \mathbb{N}$, $|F_n| = |F|$. So, by using the Bounded Convergence Theorem, we have the desired result. □

LEMMA 3.2. *Let $\varphi(t)$ be of bounded variation function on $[0,T]$ and let $x_0(t) = \int_0^t \varphi(s)db(s)$. Then $x_0 \in C_{a,b}[0,T]$ and we have*

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{(x_0(t_j) - x_0(t_{j-1}))^2}{b(t_j) - b(t_{j-1})} = \int_0^T \varphi^2(s)db(s) \tag{3.8}$$

and

(3.9)

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{((x(t_j)-a(t_j))-(x(t_{j-1})-a(t_{j-1})))(x_0(t_j)-x_0(t_{j-1}))}{b(t_j)-b(t_{j-1})}$$
$$=\int_0^T\varphi(s)dx(s)-\int_0^T\varphi(s)da(s).$$

Proof. Since $\varphi\in BV[0,T]$, $x_0(t)=\int_0^t\varphi(s)db(s)$ is absolutely continuous on $[0,T]$. Observe that

$$\sum_{j=1}^{n}\frac{(x_0(t_j)-x_0(t_{j-1}))^2}{b(t_j)-b(t_{j-1})}$$
$$=\sum_{j=1}^{n}\left(\frac{x_0(t_j)-x_0(t_{j-1})}{b(t_j)-b(t_{j-1})}\right)^2(b(t_j)-b(t_{j-1}))$$

By using the Cauchy's Mean Value Theorem in the above equation, we have

$$\sum_{j=1}^{n}\frac{(x_0(t_j)-x_0(t_{j-1}))^2}{b(t_j)-b(t_{j-1})}=\sum_{j=1}^{n}\varphi^2(\xi_j)(b(t_j)-b(t_{j-1})) \tag{3.10}$$

where $\xi_j\in[t_{j-1},t_j]$ for each $j=1,2,\cdots,n$. Similarly, we have

(3.11)

$$\begin{aligned}
&\sum_{j=1}^{n}\frac{((x(t_j)-a(t_j))-(x(t_{j-1})-a(t_{j-1})))(x_0(t_j)-x_0(t_{j-1}))}{b(t_j)-b(t_{j-1})}\\
&=\sum_{j=1}^{n}\left(\frac{x_0(t_j)-x_0(t_{j-1})}{b(t_j)-b(t_{j-1})}\right)(x(t_j)-x(t_{j-1}))\\
&\qquad-\sum_{j=1}^{n}\left(\frac{x_0(t_j)-x_0(t_{j-1})}{b(t_j)-b(t_{j-1})}\right)(a(t_j)-a(t_{j-1}))\\
&=\sum_{j=1}^{n}\varphi(\xi_j)(x(t_j)-x(t_{j-1}))-\sum_{j=1}^{n}\varphi(\xi_j)(a(t_j)-a(t_{j-1})).
\end{aligned}$$

Hence equations (3.10) and (3.11) converge to the followings

$$\int_0^T \varphi^2(s)db(s) \ \text{ and } \ \int_0^T \varphi(s)dx(s) - \int_0^T \varphi(s)da(s)$$

as $n \to \infty$, respectively. Thus we have the desired results. $\square$

THEOREM 3.3. *Let φ and x_0 be given as in Lemma 3.2 and let F be a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional. Then $F(x+x_0)$ is $\mathcal{B}(C_{a,b}[0,T])$-measurable and*

$$\int_{C_{a,b}[0,T]} F(y)d\mu(y) = \int_{C_{a,b}[0,T]} F(x+x_0)J(x,x_0)d\mu(x) \tag{3.12}$$

where

$$J(x,x_0) = \exp\biggl\{ -\frac{1}{2}\int_0^T \varphi^2(s)db(s) + \int_0^T \varphi(s)da(s) - \int_0^T \varphi(s)dx(s) \biggr\}. \tag{3.13}$$

Proof. It suffices to show the case in which the functional F is bounded on $C_{a,b}[0,T]$. Let us first consider the case F is bounded, continuous, and $F(y) = 0$ for any $y \in \{x \in C_{a,b}[0,T] : \|x\| > M\}$ where $M > 0$. Then $F(x+x_0)$ is measurable and there exists a positive real number K such that $|F(x)| \le K$ for all $x \in C_{a,b}[0,T]$.

Let $\tau : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0,T]$. Define a function G on $\mathbb{R}^n$ by $G(\vec{\xi}) = F([\vec{\xi}]_n)$ for each $\vec{\xi} \in \mathbb{R}^n$. Then G is bounded and continuous. Hence we see that

$$F([y]_n) = G(y(t_1), \cdots, y(t_n)). \tag{3.14}$$

By using (3.14) and the Change of Variables Theorem, we have

$$\begin{aligned} &\int_{C_{a,b}[0,T]} F([y]_n)d\mu(y) \\ &= \int_{C_{a,b}[0,T]} G(y(t_1), \cdots, y(t_n))d\mu(y) \\ &= \int_{\mathbb{R}^n} \Bigl((2\pi)^n \prod_{j=1}^n (b(t_j) - b(t_{j-1}))\Bigr)^{-1/2} G(v_1, \cdots, v_n) \\ &\qquad \cdot \exp\biggl\{ -\frac{1}{2}\sum_{j=1}^n \frac{((v_j - a(t_j)) - (v_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})} \biggr\} d\vec{v}. \end{aligned} \tag{3.15}$$

Let $\beta_j = x_0(t_j)$ and $u_j = v_j - \beta_j$. Then we see that

$$\begin{aligned} F([x+x_0]_n) &= G(x(t_1)+x_0(t_1), \cdots, x(t_n)+x_0(t_n)) \\ &= G(x(t_1)+\beta_1, \cdots, x(t_n)+\beta_n)) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &((v_j - a(t_j)) - (v_{j-1} - a(t_{j-1})))^2 \\ &= ((u_j - a(t_j)) - (u_{j-1} - a(t_{j-1})))^2 + (\beta_j - \beta_{j-1})^2 \\ &\qquad + 2(\beta_j - \beta_{j-1})((u_j - a(t_j)) - (u_{j-1} - a(t_{j-1}))). \end{aligned} \tag{3.17}$$

By applying (3.16) and (3.17) above to the last equation of (3.15), we have

$$\begin{aligned} &\int_{C_{a,b}[0,T]} F([y]_n) d\mu(y) \\ &= \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(x_0(t_j) - x_0(t_{j-1}))^2}{b(t_j) - b(t_{j-1})} \right\} \int_{C_{a,b}[0,T]} F([x+x_0]_n) \\ &\quad \cdot \exp\left\{ - \sum_{j=1}^{n} \frac{((x(t_j) - a(t_j)) - (x(t_{j-1}) - a(t_{j-1})))}{b(t_j) - b(t_{j-1})} \right. \\ &\qquad\qquad \left. \cdot (x_0(t_j) - x_0(t_{j-1})) \right\} d\mu(x). \end{aligned} \tag{3.18}$$

Assume that $x + x_0 \notin \{x \in C_{a,b}[0,T] : \|y\| > M\}$. Then we see that

$$\|[x+x_0]_n\| \leq \|x + x_0\| \leq M$$

and so we have

$$|x(t_j)| \leq |x_0(t_j)| + |x(t_j) + x_0(t_j)| \leq \|x_0\| + M \tag{3.19}$$

for any $x \in C_{a,b}[0,T]$. By using (3.11) and (3.19), we obtain that

$$
\begin{aligned}
&\left|\sum_{j=1}^{n} \frac{((x(t_j)-a(t_j))-(x(t_{j-1})-a(t_{j-1})))(x_0(t_j)-x_0(t_{j-1}))}{b(t_j)-b(t_{j-1})}\right| \\
&= \left|\sum_{j=1}^{n} \varphi(\xi_j)(x(t_j)-x(t_{j-1})) - \sum_{j=1}^{n} \varphi(\xi_j)(a(t_j)-a(t_{j-1}))\right| \\
&\le \left|\sum_{j=1}^{n} \varphi(\xi_j)(x(t_j)-x(t_{j-1}))\right| + \left|\sum_{j=1}^{n} \varphi(\xi_j)(a(t_j)-a(t_{j-1}))\right| \\
&= \left|\varphi(\xi_n)x(t_n) - \sum_{j=1}^{n}(\varphi(\xi_j)-\varphi(\xi_{j-1}))x(t_{j-1})\right| \\
&\qquad\qquad + \left|\varphi(\xi_n)a(t_n) - \sum_{j=1}^{n}(\varphi(\xi_j)-\varphi(\xi_{j-1}))a(t_{j-1})\right| \\
&\le |\varphi(\xi_n)||x(t_n)| + \sum_{j=1}^{n} |\varphi(\xi_j)-\varphi(\xi_{j-1})||x(t_{j-1})| \\
&\qquad\qquad + |\varphi(\xi_n)||a(t_n)| + \sum_{j=1}^{n} |\varphi(\xi_j)-\varphi(\xi_{j-1})||a(t_{j-1})| \\
&\le (\|x_0\| + M + \|a\|)(\|\varphi\| + V_0^T(\varphi))
\end{aligned}
$$

where $V_0^T(\varphi)$ is the total variation of φ on $[0,T]$. So the integrand of (3.18) is bounded by the following

$$K \exp\{(\|x_0\| + M + \|a\|)(\|\varphi\| + V_0^T(\varphi))\}.$$

Thus, by using the Bounded Convergence Theorem, (3.8), and (3.9), the expression (3.17) converges to

$$
\begin{aligned}
&\exp\left\{-\frac{1}{2}\int_0^T \varphi^2(s)db(s)\right\} \\
&\cdot \int_{C_{a,b}[0,T]} F(x+x_0)\exp\left\{-\int_0^T \varphi(s)dx(s) + \int_0^T \varphi(s)da(s)\right\}d\mu(x).
\end{aligned}
$$

Hence by using Lemma 2.1, we obtain the equation (3.12) above.

Now, let F be a nonnegative, bounded and continuous functional on $C_{a,b}[0,T]$. For each $n \in \mathbb{N}$, define the function M_n by

$$M_n(u) = \begin{cases} 1 & (0 \le u \le n) \\ n+1-u & (n \le u \le n+1) \\ 0 & (n+1 \le u). \end{cases}$$

Then M_n is a continuous real valued function. Let $F_n(x) = F(x)M_n(\|x\|)$. Then F_n satisfies the hypothesis of the first case. So proceeding as in the proof above, we obtain

$$\begin{aligned}\int_{C_{a,b}[0,T]} F_n(y)d\mu(y) = \exp&\Big\{-\frac{1}{2}\int_0^T \varphi^2(s)db(s) + \int_0^T \varphi(s)da(s)\Big\} \\ &\cdot \int_{C_{a,b}[0,T]} F_n(x+x_0)\exp\Big\{-\int_0^T \varphi(s)dx(s)\Big\}d\mu(x).\end{aligned}$$

Since $\{F_n\}$ is a monotone increasing sequence of functionals, $F_n \to F$ as $n \to \infty$. Hence by using the Monotone Convergence Theorem, we have the desired result. $\square$

THEOREM 3.4. *Let φ, x_0, and F be given as in Theorem 3.3. Then*

$$\begin{aligned}\int_{C_{a,b}[0,T]} F(x+x_0)d\mu(x) = \exp&\Big\{-\frac{1}{2}\int_0^T \varphi^2(s)db(s) - \int_0^T \varphi(s)da(s)\Big\} \\ &\cdot \int_{C_{a,b}[0,T]} F(x)\exp\Big\{\int_0^T \varphi(s)dx(s)\Big\}d\mu(x).\end{aligned}$$

Proof. Let $G(x) = F(x)\exp\{\int_0^T \varphi(s)dx(s)\}$. Then by using equation (2.12), we have

$$
\begin{aligned}
&\int_{C_{a,b}[0,T]} F(x)\exp\Big\{\int_0^T \varphi(s)dx(s)\Big\}d\mu(x)\\
&= \int_{C_{a,b}[0,T]} G(x)d\mu(x)\\
&= \int_{C_{a,b}[0,T]} G(x+x_0)J(x,x_0)d\mu(x)\\
&= \int_{C_{a,b}[0,T]} F(x+x_0)\exp\Big\{\int_0^T \varphi(s)d(x(s)+x_0(s))\Big\}\\
&\quad\cdot\exp\Big\{-\frac{1}{2}\int_0^T \varphi^2(s)db(s) + \int_0^T \varphi(s)da(s) - \int_0^T \varphi(s)dx(s)\Big\}d\mu(x)\\
&= \exp\Big\{\frac{1}{2}\int_0^T \varphi^2(s)db(s) + \int_0^T \varphi(s)da(s)\Big\}\int_{C_{a,b}[0,T]} F(x+x_0)d\mu(x).
\end{aligned}
$$

Hence we have the desired result. □

4. The general translation theorem

In this section we consider the general translation theorem for functionals on $C_{a,b}[0,T]$.

For $u,v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t)+|a|(t)].$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space.

Let $\{e_j\}_{j=1}^\infty$ be a complete orthogonal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(e_j,e_k)_{a,b} = \begin{cases} 0 & , \ j \neq k \\ 1 & , \ j = k \end{cases},$$

and for each $v \in L^2_{a,b}[0,T]$, let

$$v_n(t) = \sum_{j=1}^{n} (v, e_j)_{a,b} e_j(t)$$

for $n = 1, 2, \cdots$. Then for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund(PWZ) stochastic integral $\langle v, x\rangle$ is defined by the formula

$$\langle v, x\rangle = \lim_{n\to\infty} \int_0^T v_n(t)dx(t)$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists.

REMARK 4.1. Following are some facts about the PWZ stochastic integral

(i) For each $v \in L^2_{a,b}[0,T]$, the PWZ integral $\langle v, x\rangle$ exists for μ-a.e. $x \in C_{a,b}[0,T]$.

(ii) The PWZ integral $\langle v, x\rangle$ is essentially independent of the complete orthonormal set $\{e_j\}_{j=1}^{\infty}$.

(iii) If $v \in BV[0,T]$, then PWZ integral $\langle v, x\rangle$ equals the Riemann -Stieltjes integral $\int_0^T v(s)dx(s)$ for s-a.e. $x \in C_{a,b}[0,T]$.

(iv) The PWZ integral has the expected linearity properties.

LEMMA 4.1. *Let $\varphi_n \in C_{a,b}[0,T] \cap BV[0,T]$ for each $n \in \mathbb{N}$ and let φ_n converge in the space $L^2_{a,b}[0,T]$ as $n \to \infty$. Then for any real number λ, $\exp\{\lambda \int_0^T \varphi_n(t)dx(t)\}$ converges in the space $L^2(C_{a,b}[0,T])$ as $n \to \infty$.*

Proof. The proof given in [3] with the current hypotheses on $a(t)$ and $b(t)$ also works here. □

Now, we obtain the general translation theorem of a functional on $C_{a,b}[0,T]$.

THEOREM 4.2. *Let $\varphi(t) \in L^2_{a,b}[0,T]$ and let $x_0(t) = \int_0^t \varphi(s)db(s)$. If F be a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional on $C_{a,b}[0,T]$, then $F(x+x_0)$ is $\mathcal{B}(C_{a,b}[0,T])$-measurable and*

$$\begin{aligned}\int_{C_{a,b}[0,T]} F(y)d\mu(y) &= \exp\Big\{ -\frac{1}{2}(\varphi^2, b') + (\varphi, a')\Big\} \\ &\quad\cdot \int_{C_{a,b}[0,T]} F(x+x_0)\exp\{-\langle\varphi, x\rangle\}d\mu(x)\end{aligned} \tag{4.1}$$

where

$$(\varphi^2, b') = \int_0^T \varphi^2(t)b'(t)dt = \int_0^T \varphi(t)db(t)$$

and

$$(\varphi, a') = \int_0^T \varphi(t)a'(t)dt = \int_0^T \varphi(t)da(t).$$

Proof. It suffices to show the case in which the functional F is bounded and continuous on $C_{a,b}[0,T]$. Let $\{e_j\}_{j=1}^\infty$ be complete orthonormal set in $L^2_{a,b}[0,T]$ with $e_j \in C_{a,b}[0,T] \cap BV[0,T]$ for each $j \in \mathbb{N}$. For each $n \in \mathbb{N}$, let

$$\varphi_n(t) = \sum_{j=1}^{n}(\varphi, e_j)_{a,b}e_j(t). \tag{4.2}$$

Then $\varphi_n \in C_{a,b}[0,T] \cap BV[0,T]$ and

$$\|\varphi_n - \varphi\|_{a,b} \longrightarrow 0 \tag{4.3}$$

as $n \to \infty$. For each $n \in \mathbb{N}$, define

$$x_{0,n}(t) = \int_0^t \varphi_n(s)db(s). \tag{4.4}$$

Then, by using equation (3.12), we see that

(4.5)
$$\begin{aligned}
&\int_{C_{a,b}[0,T]} F(y)d\mu(y) \\
&= \exp\bigg\{-\frac{1}{2}\int_0^T \varphi_n^2(s)db(s) + \int_0^T \varphi_n(s)da(s)\bigg\} \\
&\qquad\qquad \cdot \int_{C_{a,b}[0,T]} F(x + x_{0,n}) \exp\bigg\{-\int_0^T \varphi_n(s)dx(s)\bigg\}d\mu(x).
\end{aligned}$$

By using Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|x_0(t) - x_{0,n}(t)| &= \left| \int_0^t (\varphi(s) - \varphi_n(s)) db(s) \right| \\
&\le \int_0^T \big| (\varphi(s) - \varphi_n(s)) \chi_{[0,t]}(s) \big| d[b(s) + |a|(s)] \\
&\le \|\varphi - \varphi_n\|_{a,b} \sqrt{b(t) + |a|(t)} \\
&\le \|\varphi - \varphi_n\|_{a,b} \sqrt{b(T) + |a|(T)}.
\end{aligned} \tag{4.6}
$$

Hence by using (4.6) and (4.3), we obtain that

$$
\|x_0 - x_{0,n}\| \longrightarrow 0. \tag{4.7}
$$

Thus for all $x \in C_{a,b}[0,T]$

$$
F(x + x_{0,n}) \longrightarrow F(x + x_0). \tag{4.8}
$$

Since F is bounded, by applying the Bounded Convergence Theorem, we have

$$
\int_{C_{a,b}[0,T]} |F(x + x_{0,n}) - F(x + x_0)|^2 d\mu(x) \longrightarrow 0. \tag{4.9}
$$

Note that

$$
\int_0^T \varphi_n(t) dx(t) \longrightarrow \langle \varphi, x \rangle \quad \mu - \text{a.e.} x \in C_{a,b}[0,T].
$$

So by using (4.3) and Lemma 4.1 with $\lambda = -1$

$$
\exp\left\{ - \int_0^T \varphi_n(t) dx(t) \right\} \longrightarrow \exp\{-\langle \varphi, x \rangle\}
$$

in the space $L^2(C_{a,b}[0,T])$. Further, by equation (4.9) above, we obtain

$$
\begin{aligned}
&\int_{C_{a,b}[0,T]} F(x + x_{0,n}) \exp\left\{ - \int_0^T \varphi_n(t) dx(t) \right\} d\mu(x) \\
&\qquad \longrightarrow \int_{C_{a,b}[0,T]} F(x + x_0) \exp\{-\langle \varphi, x \rangle\} d\mu(x).
\end{aligned} \tag{4.10}
$$

Hence by using equations (4.5) and (4.10) we have the desired equation (4.1). □

We next use Theorem 4.2 to evaluate a translation theorem of functionals on $C_{a,b}[0,T]$.

THEOREM 4.3. *Let φ, x_0, and F be given as in Theorem 4.2. Then*

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} F(x+x_0)d\mu(x) \\
&\quad = \exp\left\{-\frac{1}{2}(\varphi^2, b') - (\varphi, a)\right\} \int_{C_{a,b}[0,T]} F(x)\exp\{\langle\varphi, x\rangle\}d\mu(x).
\end{aligned}$$

Proof. Let $G(x) = F(x)\exp\{\int_0^T \varphi(s)dx(s)\}$. Then, proceeding as in the proof of Theorem 3.4, we have the desired result. □

References

[1] R.H. Cameron and W.T. Martin, *Translations of Wiener integrals under translations*, Annals of Math. **45(2)**, 386-396.
[2] R.H. Cameron and R.E. Graves, *Additive functionals on a space of continuous functions I*, Trans. Amer. Math. Soc. **70**, 160-176.
[3] S.J. Chang and D.M. Chung, *Conditional function space integrals with applications*, Rocky Mountain J. Math. **26. 1**, 37-62.
[4] S.J. Chang, C. Park and D. Skoug, *Translation theorems for Fourier-Feynman transforms and conditional Fourier-Feynman transforms*, Rocky Mountain J. of Math. **30(2)**, 477-496.
[5] G.W. Johnson and D. Skoug, *Notes on the Feynman integral I*, Pacific J. Math. **93**, 313-324.
[6] ______, *Notes on the Feynman integral II*, J. Func. Anal. **41**, 277-289.
[7] ______, *Notes on the Feynman integral III : The Schrödinger equation*, Pacific J. Math. **105**, 321-358.
[8] J. Yeh, *Stochastic Processes and the Wiener Integral*, Marcel Dekker, Inc., New York, 1973.

Department of Mathematics
Dankook University
Cheonan 330-714, Korea
E-mail: jgchoi@dankook.ac.kr

Department of Mathematics
Dankook University
Cheonan 330-714, Korea
E-mail: yseo@dankook.ac.kr