KYUNGPOOK Math. J. 43(2003), 605-616

How to Characterize Equalities for the Generalized Inverse $A^{(2)}_{T,S}$ of a Matrix

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Abstract. In this paper, some rank equalities related to generalized inverses $A^{(2)}_{T,S}$ of a matrix are presented. As applications, a variety of rank equalities related to the M-P inverse, the Drazin inverse, the group inverse, the weighted M-P inverse, the Bott-Duffin inverse and the generalized Bott-Duffin inverse are established.

1. Introduction

In the theory of generalized inverses of matrices, rank equalities related to generalized inverses are the important subjects, and have been widely studied ([3], [5]-[10]). How to characterize equalities for the M-P inverse of a matrix? Tian ([6]) presented a simple and excellent method for coping with above problem, and use it to characterize a variety of valuable equalities related to the M-P inverse of a matrix. It is well-known that M-P inverse $A^{\dagger}$ is a generalized inverse $A^{(2)}_{T,S}$, and it is also well-known that the Drazin inverse A^D, the weighted M-P inverse $A^{\dagger}_{M,N}$, the group inverse A_g, the Bott-Duffin inverse $A^{(-1)}_{(L)}$ and the generalized Bott-Duffin inverse $A^{(+)}_{(L)}$ are all generalized inverses $A^{(2)}_{T,S}$. So, it is significant to study the rank equalities related to generalized inverse $A^{(2)}_{T,S}$. Following [6], in this paper, we present a variety of rank equalities related to the generalized inverse $A^{(2)}_{T,S}$. As their applications, we shall give some rank equalities related to $A^{\dagger}, A^D, A^{\dagger}_{M,N}, A_g, A^{(-1)}_{(L)}$ and $A^{(+)}_{(L)}$. The matrices considered in this paper are over the field C of complex numbers. For $A \in C^{m\times n}$, we use $A^*, r(A), R(A)$ and $N(A)$ to stand for the conjugate transpose, the rank, the range and the null space of A, respectively.

Received December 30, 2002.
2000 Mathematics Subject Classification: 15A03, 15A09.
Key words and phrases: rank, generalized inverse $A^{(2)}_{T,S}$.
This work was partly Supported by Shanghai Priority Academic Discipline Foundation, Shanghai, China.

Lemma 1.1 ([1]). *Let $A \in C^{m\times n}$ be of rank r, let T be a subspace of C^n of dimension $s \le r$, and let S be a subspace of C^m of dimension $m-s$. Then A has a $\{2\}-$ inverse X such that $R(X)=T$ and $N(X)=S$ if and only if*

$$AT \oplus S = C^m, \tag{1.1}$$

in which case X is unique, this X is denoted by $A^{(2)}_{T,S}$.

Lemma 1.2 ([11]). *Let $A \in C^{m\times n}$ be of rank r, and let T be a subspace of C^n of dimension $s \le r$, and let S be a subspace of C^m of dimension $m-s$. In addition, suppose $G \in C^{n\times m}$ such that $R(G)=T$ and $N(G)=S$. If, A has a $\{2\}-$ inverse $A^{(2)}_{T,S}$, then*

$$\mathrm{Ind}(AG) = \mathrm{Ind}(GA) = 1. \tag{1.2}$$

Further, we have

$$A^{(2)}_{T,S} = G(AG)_g = (GA)_g G. \tag{1.3}$$

From [11] and Lemma 1.2, let G be equal to $A^*, N^{-1}A^*M, A^k, A, P_L$ and P_S respectively, we have

Lemma 1.3 ([1], [2], [11]).

(1) *Let $A \in C^{m\times n}$, then one has*

$$A^\dagger = A^{(2)}_{R(A^*),N(A^*)} = A^*(AA^*)_g = (A^*A)_g A^*,$$
$$A^\dagger_{M,N} = A^{(2)}_{R(N^{-1}A^*M),N(N^{-1}A^*M)} = N^{-1}A^*M(AN^{-1}A^*M)_g = (N^{-1}A^*MA)_g N^{-1}A^*M;$$

where M, N are Hermitian positive matrices of order m and n, respectively;

(2) *Let $A \in C^{n\times n}$, then one has*

$$A^D = A^{(2)}_{R(A^k),N(A^k)} = A^k(A^{k+1})_g = (A^{k+1})_g A^k,$$
$$A_g = A^{(2)}_{R(A),N(A)} = A(A^2)_g = (A^2)_g A,$$
$$A_g = A(A^3)^\dagger A;$$

(3) *Let $A \in C^{n\times n}$, then one has*

$$A^{(-1)}_{(L)} = A^{(2)}_{L,L^\perp} = P_L(AP_L)_g = (P_LA)_g P_L,$$

where L is a subspace of C^n satisfying $AL \oplus L^\perp = C^n$;

$$A^{(\dagger)}_{(L)} = A^{(2)}_{S,S^\perp} = P_S(AP_S)_g = (P_SA)_g P_S,$$

where L is a subspace of C^n, $S = R(P_LA)$ and A is an L-p.s.d matrix, i.e. A is a Hermitian matrix with the properties: P_LAP_L is nonnegative definite, and $N(P_LAP_L) = N(AP_L)$.

Lemma 1.4 ([3]). *Let $A \in C^{m\times n}$, $B \in C^{m\times k}$ and $C \in C^{l\times n}$ be given, and suppose that*

$$R(AQ) = R(A), \quad R[(PA)^*] = R(A^*),$$

then

$$r(AQ, B) = r(A, B), \quad r\begin{pmatrix} PA \\ C \end{pmatrix} = r\begin{pmatrix} A \\ C \end{pmatrix}.$$

Lemma 1.5 ([6]). *Let $A \in C^{m\times n}$, $B \in C^{m\times k}$, $C \in C^{l\times n}$ and $D \in C^{l\times k}$ be given. Then we have*

$$r(D - CA^\dagger B) = r\begin{pmatrix} A^*AA^* & A^*B \\ CA^* & D \end{pmatrix} - r(A). \tag{1.4}$$

Furthermore, let

$$C = (C_1, C_2), \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

then (1.4) *becomes*

$$r(D - C_1A_1^\dagger B_1 - C_2A_2^\dagger B_2) = r\begin{pmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 \\ 0 & A_2^*A_2A_2^* & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D \end{pmatrix} - r(A_1) - r(A_2). \tag{1.5}$$

In particular, if

$$R(B_1) \subseteq R(A_1), \quad R(C_1^*) \subseteq R(A_1^*), \quad R(B_2) \subseteq R(A_2), \quad R(C_2^*) \subseteq R(A_2^*),$$

then

$$r(D - C_1A_1^\dagger B_1 - C_2A_2^\dagger B_2) = r\begin{pmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{pmatrix} - r(A_1) - r(A_2). \tag{1.6}$$

Lemma 1.6 ([5], [6]). *Let $A \in C^{m\times n}$ be given, and let $P \in C^{m\times m}$ and $Q \in C^{n\times n}$ be two idempotent matrices. Then*

$$r(PA - AQ) = r\begin{pmatrix} PA \\ Q \end{pmatrix} + r(AQ, P) - r(P) - r(Q),$$
$$r(P - Q) = r\begin{pmatrix} P \\ Q \end{pmatrix} + r(Q, P) - r(P) - r(Q).$$

2. The rank equalities related to generalized inverse $A^{(2)}_{T,S}$ of a matrix

In this section, some rank equalities related to generalized inverses $A^{(2)}_{T,S}$ of a matrix are given.

Theorem 2.1. *Let $A \in C^{m\times m}$ be given. Suppose that $G \in C^{m\times m}$ such that $R(G) = T$ and $N(G) = S$. If A has a $\{2\}-$ inverse $A^{(2)}_{T,S}$*

$$A^{(2)}_{T,S} = G(AG)_g = (GA)_g G,$$

then

$$r(AA^{(2)}_{T,S} - A^{(2)}_{T,S}A) = r\begin{pmatrix} AG \\ GA \end{pmatrix} + r(AG, GA) - 2r(AG). \tag{2.1}$$

In particular,

$$AA^{(2)}_{T,S} = A^{(2)}_{T,S}A \Longleftrightarrow R(AG) = R(GA), R[(AG)^*] = R[(GA)^*]. \tag{2.2}$$

Proof. Note that $AA^{(2)}_{T,S}$ and $A^{(2)}_{T,S}A$ are idempotent matrices. By Lemma 1.6, we first obtain

(2.3)

$$r(AA^{(2)}_{T,S} - A^{(2)}_{T,S}A) = r\begin{pmatrix} AA^{(2)}_{T,S} \\ A^{(2)}_{T,S}A \end{pmatrix} + r(AA^{(2)}_{T,S}, A^{(2)}_{T,S}A) - r(AA^{(2)}_{T,S}) - r(A^{(2)}_{T,S}A).$$

Note that

$$AA^{(2)}_{T,S} = AG(AG)_g = (AG)_g AG, \quad A^{(2)}_{T,S}A = (GA)_g GA = GA(GA)_g,$$
$$r[AG(AG)_g] = r(AG), \quad r[(AG)_g AG]^* = r(AG)^*,$$

then applying Lemma 1.4,

$$r\begin{pmatrix} AA^{(2)}_{T,S} \\ A^{(2)}_{T,S}A \end{pmatrix} = r\begin{pmatrix} AG \\ GA \end{pmatrix}, \quad r(AA^{(2)}_{T,S}, A^{(2)}_{T,S}A) = r(AG, GA),$$

$$r(AA^{(2)}_{T,S}) = r(AG),$$

$$r(A^{(2)}_{T,S}A) = r[(GA)_g GA] = r(GA) = r(GA)^2 = r(GAG) = r(AG).$$

Thus (2.3) reduces to (2.1). Note that

$$r\begin{pmatrix} AG \\ GA \end{pmatrix} = r(AG) \Longleftrightarrow R(AG)^* = R(GA)^*,$$

$$r(AG, GA) = r(AG) \Longleftrightarrow R(AG) = R(GA),$$

thus (2.1) reduces to (2.2). □

Corollary 2.2. *Let $A \in C^{m\times m}$, M, N be Hermitian positive definite matrices of order m. Then*

(1) $r(AA^{\dagger} - A^{\dagger}A) = 2r(A, A^*) - 2r(A)$;
$AA^{\dagger} = A^{\dagger}A \Leftrightarrow r(A, A^*) = r(A) \Leftrightarrow A$ *is EP;*

(2) $r(AA^{\dagger}_{M,N} - A^{\dagger}_{M,N}A) = r(A^*, MA) + r(A^*, NA) - 2r(A)$;
$AA^{\dagger}_{M,N} = A^{\dagger}_{M,N}A \Leftrightarrow R(MA) = R(NA) = R(A^*) \Leftrightarrow$ *both* MA *and* NA *are EP;*

(3) $r(AA^{(-1)}_{(L)} - A^{(-1)}_{(L)}A) = r\begin{pmatrix} AP_L \\ P_LA \end{pmatrix} + r(AP_L, P_LA) - 2r(AP_L)$;
$AA^{(-1)}_{(L)} = A^{(-1)}_{(L)}A \Leftrightarrow R(AP_L) = R(P_LA), R(AP_L)^* = R(P_LA)^*$;

(4) $r(AA^{(+)}_{(L)} - A^{(+)}_{(L)}A) = r\begin{pmatrix} AP_S \\ P_SA \end{pmatrix} + r(AP_S, P_SA) - 2r(AP_S)$;
$AA^{(-1)}_{(L)} = A^{(-1)}_{(L)}A \Leftrightarrow R(AP_S) = R(P_SA), R(AP_S)^* = R(P_SA)^*$.

Theorem 2.3. *Let* $A \in C^{m\times m}$ *be given and* k *be an integer with* $k \geq 2$. *Suppose that* $G \in C^{m\times m}$ *such that* $R(G) = T$ *and* $N(G) = S$. *If* A *has a* $\{2\}-$ *inverse* $A^{(2)}_{T,S}$

$$A^{(2)}_{T,S} = G(AG)_g = (GA)_gG,$$

then

$$r(A^kA^{(2)}_{T,S} - A^{(2)}_{T,S}A^k) = r\begin{pmatrix} AG \\ GA^k \end{pmatrix} + r(A^kG, GA) - 2r(AG). \tag{2.4}$$

In particular,

$$A^kA^{(2)}_{T,S} = A^{(2)}_{T,S}A^k \Longleftrightarrow R(A^kG) \subseteq R(GA), R[(GA^k)^*] \subseteq R[(AG)^*]. \tag{2.5}$$

Proof. Writing $A^kA^{(2)}_{T,S} - A^{(2)}_{T,S}A^k = -[(A^{(2)}_{T,S}A)A^{k-1} - A^{k-1}(AA^{(2)}_{T,S})]$ and applying Lemma 1.6 to it, we obtain
(2.6)

$$r(A^kA^{(2)}_{T,S} - A^{(2)}_{T,S}A^k) = r\begin{pmatrix} A^{(2)}_{T,S}A^k \\ AA^{(2)}_{T,S} \end{pmatrix} + r(A^kA^{(2)}_{T,S}, A^{(2)}_{T,S}A) - r(AA^{(2)}_{T,S}) - r(A^{(2)}_{T,S}A).$$

Note that

$$r[(GA)_gGAA^{k-1}] = r(GAA^{k-1}), \quad r[A^{k-1}AG(AG)_g] = r(A^{k-1}AG).$$

By applying Lemma 1.4, we have

$$r\begin{pmatrix} A^{(2)}_{T,S}A^k \\ AA^{(2)}_{T,S} \end{pmatrix} = r\begin{pmatrix} GA^k \\ AG \end{pmatrix}, \quad r(A^kA^{(2)}_{T,S}, A^{(2)}_{T,S}A) = r(A^kG, GA).$$

Thus (2.6) reduces to (2.4). The result in (2.5) follows immediately from (2.4). □

Corollary 2.4. *Let $A \in C^{m\times m}$, M, N be Hermitian positive definite matrices of order m. Then*

(1) $r(A^kA^\dagger - A^\dagger A^k) = r\begin{pmatrix} A^k \\ A^* \end{pmatrix} + r(A^k, A^*) - 2r(A)$;
$A^kA^\dagger = A^\dagger A^k \Leftrightarrow R(A^k) \subseteq R(A^*), \quad R[(A^k)^*] \subseteq R(A)$;

(2) $r(A^kA^\dagger_{M,N} - A^\dagger_{M,N}A^k) = r\begin{pmatrix} A^k \\ A^*M \end{pmatrix} + r(A^k, N^{-1}A^*) - 2r(A)$;
$A^kA^\dagger_{M,N} = A^\dagger_{M,N}A^k \Leftrightarrow R(A^k) \subseteq R(N^{-1}A^*), \quad R[(A^k)^*] \subseteq R(MA)$;

(3) $r(A^kA^{(-1)}_{(L)} - A^{(-1)}_{(L)}A^k) = r\begin{pmatrix} AP_L \\ P_LA^k \end{pmatrix} + r(A^kP_L, P_LA) - 2r(AP_L)$;
$A^kA^{(-1)}_{(L)} = A^{(-1)}_{(L)}A^k \Leftrightarrow R(A^kP_L) \subseteq R(P_LA), \quad R[(P_LA^k)^*] \subseteq R[(AP_L)^*]$;

(4) $r(A^kA^{(+)}_{(L)} - A^{(+)}_{(L)}A^k) = r\begin{pmatrix} AP_S \\ P_SA^k \end{pmatrix} + r(A^kP_S, P_SA) - 2r(AP_S)$;
$A^kA^{(+)}_{(L)} = A^{(+)}_{(L)}A^k \Leftrightarrow R(A^kP_S) \subseteq R(P_SA), R[(P_SA^k)^*] \subseteq R[(AP_S)^*]$.

Theorem 2.5. *Let $A \in C^{m\times m}$ be given, Suppose that $G \in C^{m\times m}$ such that $R(G) = T$ and $N(G) = S$. If A has a $\{2\}-$ inverse $A^{(2)}_{T,S}$*

$$A^{(2)}_{T,S} = G(AG)_g = (GA)_gG,$$

then

(1) $r(A^*A^{(2)}_{T,S} - A^{(2)}_{T,S}A^*) = r\begin{pmatrix} AG(A^*A - AA^*)GA & 0 & AGA^* \\ 0 & 0 & AG \\ A^*GA & GA & 0 \end{pmatrix} - 2r(AG)$;

(2) *If $R(A^*GA) \subseteq R(GA)$, $R[(AGA^*)^*] \subseteq R[(AG)^*]$, then*

$$r(A^*A^{(2)}_{T,S} - A^{(2)}_{T,S}A^*) = r[AG(A^*A - AA^*)GA];$$

(3) $A^*A^{(2)}_{T,S} = A^{(2)}_{T,S}A^* \Leftrightarrow R(A^*GA) \subseteq R(GA)$, $R[(AGA^*)^*] \subseteq R[(AG)^*]$, $AGA^*AGA = AGAA^*GA$.

Proof. From Lemma 1.2 and Lemma 1.3, we have

$$A^*A^{(2)}_{T,S} - A^{(2)}_{T,S}A^* = A^*GAG[(AG)^3]^\dagger AG - GAG[(AG)^3]^\dagger AGA^*.$$

By applying formula (1.6) in Lemma 1.5 and block Gaussian elimination, we have

$$\begin{aligned}
r(A^*A^{(2)}_{T,S} - A^{(2)}_{T,S}A^*) &= r\begin{pmatrix} (AG)^3 & 0 & AG \\ 0 & (AG)^3 & AGA^* \\ -A^*GAG & GAG & 0 \end{pmatrix} - 2r[(AG)^3] \\
&= r\begin{pmatrix} 0 & 0 & AG \\ -AGA^*(AG)^2 & (AG)^3 & AGA^* \\ -A^*GAG & GAG & 0 \end{pmatrix} - 2r(AG) \\
&= r\begin{pmatrix} AG(A^*A - AA^*)GAG & 0 & AGA^* \\ 0 & 0 & AG \\ A^*GAG & GAG & 0 \end{pmatrix} - 2r(AG) \\
&= r\begin{pmatrix} AG(A^*A - AA^*)GA & 0 & AGA^* \\ 0 & 0 & AG \\ A^*GA & GA & 0 \end{pmatrix} - 2r(AG).
\end{aligned}$$

The last equality is based on Lemma 1.4, and the results in (2) and (3) follow immediately from (1). □

Corollary 2.6. *Let $A \in C^{m\times m}$, M, N be Hermitian positive definite matrices of order m. Then*

(1) $r(A^*A^\dagger - A^\dagger A^*) = r(AA^*A^2 - A^2A^*A)$;
$A^*A^\dagger = A^\dagger A^* \Leftrightarrow AA^*A^2 = A^2A^*A \Leftrightarrow A$ *is star-dagger;*

(2) $r(A^*A^\dagger_{M,N} - A^\dagger_{M,N}A^*) = r[AN^{-1}(A^*A - AA^*)MA]$;
$A^*A^\dagger_{M,N} = A^\dagger_{M,N}A^* \Leftrightarrow AN^{-1}A^*AMA = AN^{-1}AA^*MA$;

(3) $r(A^*A^D - A^DA^*) = r\begin{pmatrix} A^k(AA^* - A^*A)A^k & 0 & A^kA^* \\ 0 & 0 & A^k \\ A^*A^k & A^k & 0 \end{pmatrix} - 2r(A^k)$;
$A^*A^D = A^DA^* \Leftrightarrow R(A^*A^k) \subseteq R(A^k),\ R[A(A^k)^*] \subseteq R[(A^k)^*]$,
$A^{k+1}A^*A^k = A^kA^*A^{k+1}$, *where* $k = \mathrm{Ind}(A)$;

(4) $r(A^*A_g - A_gA^*) = r\begin{pmatrix} A(AA^* - A^*A)A & 0 & AA^* \\ 0 & 0 & A \\ A^*A & A & 0 \end{pmatrix} - 2r(A)$;
$A^*A_g = A_gA^* \Leftrightarrow A^2A^*A = AA^*A^2$ *and E is EP.*

(5) $r(A^*A^{(-1)}_{(L)} - A^{(-1)}_{(L)}A^*)$
$= r\begin{pmatrix} AP_L(AA^* - A^*A)P_LA & 0 & AP_LA^* \\ 0 & 0 & AP_L \\ A^*P_LA & P_LA & 0 \end{pmatrix} - 2r(AP_L)$;
$A^*A^{(-1)}_{(L)} = A^{(-1)}_{(L)}A^* \Leftrightarrow R(A^*P_LA) \subseteq R(P_LA),\quad R[(AP_LA^*)^*] \subseteq R[(AP_L)^*]$,
and $AP_LA^*AP_LA = AP_LAA^*P_LA$;

(6) $r(A^*A^{(+)}_{(L)} - A^{(+)}_{(L)}A^*)$

$$= r\begin{pmatrix} AP_S(AA^* - A^*A)P_SA & 0 & AP_SA^* \\ 0 & 0 & AP_S \\ A^*P_SA & P_SA & 0 \end{pmatrix} - 2r(AP_S);$$

$A^*A^{(+)}_{(L)} = A^{(+)}_{(L)}A^* \Leftrightarrow R(A^*P_SA) \subseteq R(P_SA)$, $R[(AP_SA^*)^*] \subseteq R[(AP_S)^*]$, *and* $AP_SA^*AP_SA = AP_SAA^*P_SA$.

3. The rank equalities to power of the generalized inverse $A^{(2)}_{T,S}$ of a matrix

In this section, we present some rank equalities of matrix expressions involving power of the generalized inverses $A^{(2)}_{T,S}$ of a matrix.

Theorem 3.1. *Let $A \in C^{m\times m}$ be given, Suppose that $G \in C^{m\times m}$ such that $R(G) = T$ and $N(G) = S$. If A has a $\{2\}-$ inverse $A^{(2)}_{T,S}$*

$$A^{(2)}_{T,S} = G(AG)_g = (GA)_gG,$$

then

(1) $r[I_m \pm A^{(2)}_{T,S}] = r[A(G^2 \pm GAG)A] - r(AG) + m$;

(2) $r[I_m - (A^{(2)}_{T,S})^2] = r[A(G^2 + GAG)A] + r[A(G^2 - GAG)A] - 2r(AG) + m.$

Proof. By Lemma 1.2, Lemma 1.4 and formula (1.4) in Lemma 1.5, we easily obtain

$$\begin{aligned} r(I_m - A^{(2)}_{T,S}) &= r(I_m - GAG((AG)^3)^\dagger AG) \\ &= r\begin{pmatrix} ((AG)^3)^*(AG)^3((AG)^3)^* & ((AG)^3)^*AG \\ GAG((AG)^3)^* & I \end{pmatrix} - r(AG)^3 \\ &= r\begin{pmatrix} (AG)^3 & AG \\ GAG & I \end{pmatrix} - r(AG) \\ &= r\begin{pmatrix} AGAGA & AG \\ GA & I \end{pmatrix} - r(AG) \\ &= r\begin{pmatrix} AGAGA - AG^2A & 0 \\ 0 & I \end{pmatrix} - r(AG) \\ &= r[A(G^2 - GAG)A] - r(AG) + m. \end{aligned}$$

Similarly, we can establish the other equality of (1). Next applying a well-known rank formula $r(I - A^2) = r(I + A) + r(I - A) - m$ to $I_m - (A^{(2)}_{T,S})^2$, we obtain (2).

□

Corollary 3.2. *Let $A \in C^{m\times m}$ with $\mathrm{Ind}(A) = k$, M, N be Hermitian positive definite matrices of order m. Then*

(1) $r(I_m \pm A^{\dagger}) = r(A^2 \pm AA^*A) - r(A) + m;$
$r(I_m - (A^{\dagger})^2) = r(A^2 + AA^*A) + r(A^2 - AA^*A) - 2r(A) + m;$

(2) $r(I_m \pm A^{\dagger}_{M,N}) = r(AN^{-1}MA \pm AN^{-1}A^*MA) - r(A) + m;$
$r(I_m - (A^{\dagger}_{M,N})^2) = r(AN^{-1}MA + AN^{-1}A^*MA) + r(AN^{-1}MA - AN^{-1}A^*MA) - 2r(A) + m;$

(3) $r(I_m \pm A^D) = r(A^k \pm A^{k+1}) - r(A^k) + m = r(I \pm A);$
$r(I_m - (A^D)^2) = r(I - A^2)$, *where* $k = \mathrm{Ind}(A)$;

(4) $r(I_m \pm A_g) = r(I \pm A);$
$r(I_m - (A_g)^2) = r(I - A^2);$

(5) $r(I_m \pm A^{(-1)}_{(L)}) = r[A((P_L)^2 \pm P_L A P_L)A] - r(AP_L) + m;$
$r(I_m - A^{(-1)}_{(L)}) = r[A((P_L)^2 + P_L A P_L)A] + r[A((P_L)^2 - P_L A P_L)A] - 2r(AP_L) + m;$

(6) $r(I_m \pm A^{(+)}_{(L)}) = r[A((P_S)^2 \pm P_S A P_S)A] - r(AP_S) + m;$
$r(I_m - A^{(+)}_{(L)}) = r[A((P_S)^2 + P_S A P_S)A] + r[A((P_S)^2 - P_S A P_S)A] - 2r(AP_S) + m.$

Proof. (1), (2), (5), (6) follow from Theorem 3.1. (4) follows from (3). For (3), we use a well-known results $r(p(A)q(A)) = r(p(A)) + r(q(A)) - m$, where $p(A)$ and $q(A)$ are polynomial of A. □

Theorem 3.3. *Let $A \in C^{m\times m}$ be given, Suppose that $G \in C^{m\times m}$ such that $R(G) = T$ and $N(G) = S$. If A has a $\{2\}-$ inverse $A^{(2)}_{T,S}$*

$$A^{(2)}_{T,S} = G(AG)_g = (GA)_g G,$$

then

(1) $r[A^{(2)}_{T,S} \pm (A^{(2)}_{T,S})^2] = r[A(G^2 \pm GAG)A];$

(2) $A^{(2)}_{T,S} = (A^{(2)}_{T,S})^2 \Leftrightarrow AG^2A = (AG)^2A.$

Proof. According to a well-known rank formula

$$r(A - A^2) = r(I - A) + r(A) - m,$$

we get

$$r[A^{(2)}_{T,S} \pm (A^{(2)}_{T,S})^2] = r(I_m \pm A^{(2)}_{T,S}) + r(A^{(2)}_{T,S}) - m,$$

putting

$$r(A^{(2)}_{T,S}) = r[GAG((AG)^3)^{\dagger} AG] = r(AG),$$

and Theorem 3.1(1) to them yields the two equalities in (1). The result in (2) follows from (1). □

Corollary 3.4. *Let $A \in C^{m\times m}$, M, N be Hermitian positive definite matrices of order m. Then*

(1) $r(A^\dagger \pm (A^\dagger)^2) = r(A^2 \pm AA^*A)$;
$(A^\dagger)^2 = A^\dagger \Leftrightarrow AA^*A = A^2$;

(2) $r(A^\dagger_{M,N} \pm (A^\dagger_{M,N})^2) = r(AN^{-1}MA \pm AN^{-1}A^*MA)$;
$(A^\dagger_{M,N})^2 = A^\dagger_{M,N} \Leftrightarrow AN^{-1}MA = AN^{-1}A^*MA$;

(3) $r(A^D \pm (A^D)^2) = r(A^k \pm A^{k+1})$;
$(A^D)^2 = A^D \Leftrightarrow A^{k+1} = A^k$, *where* $k = \mathrm{Ind}(A)$;

(4) $r(A_g \pm (A_g)^2) = r(A \pm A^2)$;
$(A_g)^2 = A_g \Leftrightarrow A^2 = A$, *i.e. A is idempotent;*

(5) $r(A^{(-1)}_{(L)} \pm (A^{(-1)}_{(L)})^2) = r[A((P_L)^2 \pm P_LAP_L)A]$;
$(A^{(-1)}_{(L)})^2 = A^{(-1)}_{(L)} \Leftrightarrow A(P_L)^2A = (AP_L)^2A$;

(6) $r(A^{(+)}_{(L)} \pm (A^{(+)}_{(L)})^2) = r[A((P_S)^2 \pm P_SAP_S)A]$;
$(A^{(+)}_{(L)})^2 = A^{(+)}_{(L)} \Leftrightarrow A(P_S)^2A = (AP_S)^2A$.

Theorem 3.5. *Let $A \in C^{m\times m}$ be given, Suppose that $G \in C^{m\times m}$ such that $R(G) = T$ and $N(G) = S$. If A has a $\{2\}-$ inverse $A^{(2)}_{T,S}$*

$$A^{(2)}_{T,S} = G(AG)_g = (GA)_gG,$$

then

(1) $r[A^{(2)}_{T,S} - (A^{(2)}_{T,S})^3] = r[A(G^2 + GAG)A] + r[A(G^2 - GAG)A] - r(AG)$;

(2) $A^{(2)}_{T,S} = (A^{(2)}_{T,S})^3 \Leftrightarrow r[A(G^2 + GAG)A] + r[A(G^2 - GAG)A] = r(AG)$.

Proof. Applying a well-known rank equality

$$r(A - A^3) = r(A + A^2) + r(A - A^2) - r(A),$$

we obtain

$$r[A^{(2)}_{T,S} - (A^{(2)}_{T,S})^3] = r[A^{(2)}_{T,S} + (A^{(2)}_{T,S})^2] + r[A^{(2)}_{T,S} - (A^{(2)}_{T,S})^2] - r(A^{(2)}_{T,S}).$$

Then putting Theorem 3.3(1) and $r(A^{(2)}_{T,S}) = r(AG)$ in it yields (1). The result in (2) follows from (1). □

Corollary 3.6. *Let $A \in C^{m\times m}$, M, N be Hermitian positive definite matrices of order m. Then*

(1) $r(A^\dagger - (A^\dagger)^3) = r(A^2 + AA^*A) + r(A^2 - AA^*A) - r(A)$;
$(A^\dagger)^3 = A^\dagger \Leftrightarrow r(A^2 + AA^*A) + r(A^2 - AA^*A) = r(A)$;

(2) $r(A^\dagger_{M,N} - (A^\dagger_{M,N})^3) = r(AN^{-1}MA + AN^{-1}A^*MA) + r(AN^{-1}MA - AN^{-1}A^*MA) - r(A)$;
$(A^\dagger_{M,N})^3 = A^\dagger_{M,N} \Leftrightarrow r(AN^{-1}MA + AN^{-1}A^*MA) + r(AN^{-1}MA - AN^{-1}A^*MA) = r(A)$;

(3) $r(A^D - (A^D)^3) = r(A^k + A^{k+1}) + r(A^k - A^{k+1}) - r(A^k) = r(A^k - A^{k+2})$;
$(A^D)^3 = A^D \Leftrightarrow A^{k+2} = A^k$, *where* $k = \mathrm{Ind}(A)$;

(4) $r(A_g - (A_g)^3) = r(A - A^3)$;
$(A_g)^3 = A_g \Leftrightarrow A^3 = A$, *i.e.* A *is tripotent;*

(5) $r(A^{(-1)}_{(L)} - (A^{(-1)}_{(L)})^3) = r[A((P_L)^2 + P_LAP_L)A] + r[A((P_L)^2 - P_LAP_L)A] - r(AP_L)$;
$(A^{(-1)}_{(L)})^3 = A^{(-1)}_{(L)} \Leftrightarrow r[A((P_L)^2 + P_LAP_L)A] + r[A((P_L)^2 - P_LAP_L)A] = r(AP_L)$;

(6) $r(A^{(+)}_{(L)} - (A^{(+)}_{(L)})^3) = r[A((P_S)^2 + P_SAP_S)A] + r[A((P_S)^2 - P_SAP_S)A] - r(AP_S)$;
$(A^{(+)}_{(L)})^3 = A^{(+)}_{(L)} \Leftrightarrow r[A((P_S)^2 + P_SAP_S)A] + r[A((P_S)^2 - P_SAP_S)A] = r(AP_S)$.

Proof. (1), (2), (5), (6) follow from Theorem 3.5 and Corollary 3.4. (4) follows from (3). We only prove (3). Applying a well-known rank equality

$$r(p(A)q(A)) = r(p(A)) + r(q(A)) - m,$$

where $p(A), q(A)$ are polynomial of A gives

$$\begin{aligned} r[A^D - (A^D)^3] &= r(A^k + A^{k+1}) + r(A^k - A^{k+1}) - r(A^k) \\ &= r(A^{2k} - A^{2(k+1)}) + m - r(A^k) \\ &= r(A^k) + r(A^k - A^{k+2}) - m + m - r(A^k) = r(A^k - A^{k+2}). \end{aligned}$$

□

4. Concluding remarks

In this paper, some rank equalities related to the generalized inverse $A^{(2)}_{T,S}$ of a matrix have been established and some well-known results have been extended.

Acknowledgement. The author wishes to express his sincere thanks to Professor Musheng Wei and Professor Guoliang Chen for their guidance and also to thank the referee for helpful comments and valuable suggestions.

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