

Rate of Convergence of the Integral Type Lupas-Bézier Operators

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ABSTRACT. In this paper we introduce the integral type Lupas-Bézier operator $\tilde{B}_{n,\alpha}$, which is a new approximation operator of probabilistic type. We study the rate of pointwise convergence of the operators $\tilde{B}_{n,\alpha}$ for local bounded functions and get an asymptotically estimate by means of some methods and techniques of probability theory.

1. Introduction

Rate of approximation of the Durrmeyer-Bézier operators for functions of bounded variation defined on finite interval has been studied by Zeng and Chen ([1]). In present paper we will discuss the case of the function of operators defined on infinite interval. These operators are so-called integral type Lupas-Bézier operators. Furthermore, we will consider a new class of function which is more general than the class of function of bounded variation considered in [1]. The probabilistic methods and techniques play key roles in this paper. The sign \mathbf{N} denotes the set of nonnegative integer throughout the paper.

Let $b_{nk}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$ ($k \in \mathbf{N}$) be Lepas basis functions and let $J_{nk}(x) = \sum_{i=k}^{\infty} b_{ni}(x)$ be Lupas-Bézier basis functions. For a function f defined on $[0, \infty)$ and $\alpha \geq 1$, the integral type Lupas-Bézier operators $\tilde{B}_{n,\alpha}$ are defined by

$$(1) \quad \tilde{B}_{n,\alpha}(f, x) = (n-1) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \int_0^{\infty} f(t) b_{nk}(t) dt$$

where $Q_{nk}^{(\alpha)}(x) = J_{nk}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$. When $\alpha = 1$, $\tilde{B}_{n,1}$ is just the well-known

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modified Lupas operators given by

$$(2) \quad \tilde{B}_{n,1}(f, x) = (n-1) \sum_{k=0}^{\infty} b_{nk}(x) \int_0^{\infty} f(t) b_{nk}(t) dt.$$

We consider the following class of function $\Phi_{loc,\beta}$ and the quantity $\omega_x(f, \lambda)$.

$\Phi_{loc,\beta} = \{f : f \text{ is bounded in every finite subinterval of } [0, \infty), \text{ and}$

$$f(t) = O(t^\beta) \text{ for some } \beta > 0 \text{ as } t \rightarrow \infty.\}$$

$$\omega_x(f, \lambda) = \sup_{t \in [x-\lambda, x+\lambda]} |f(t) - f(x)|.$$

It is clear that (i) $\omega_x(f, \lambda)$ is monotone non-decreasing with respect to λ . (ii) If f is bounded variation on $[a, b]$, and $\overset{a}{\underset{b}{V}}(f)$ denotes the total variation of f on $[a, b]$, then

$$\omega_x(f, \lambda) \leq \overset{x+\lambda}{\underset{x-\lambda}{V}}(f).$$

The main result of this paper is as follows:

Theorem. *Let $f \in \Phi_{loc,\beta}$ and $f(x+), f(x-)$ exists at $x \in (0, \infty)$. Then for n sufficiently large we have*

$$(3) \quad \left| \tilde{B}_{n,\alpha}(f, x) - \frac{f(x+) + \alpha f(x-)}{\alpha + 1} \right| = \frac{16\alpha(1+x) + x}{nx} \sum_{k=1}^n \omega_x(g_x, x/\sqrt{k}) \\ + \frac{8\alpha\sqrt{1+x}}{\sqrt{xn}} |f(x+) - f(x-)| + \frac{\alpha(2x)^\beta}{x^{2m}} O(n^{-[(m+1)/2]}),$$

where

$$(4) \quad g_x(t) = \begin{cases} f(t) - f(x+), & x < t < \infty \\ 0, & t = x \\ f(t) - f(x-), & 0 \leq t < x. \end{cases}$$

In particular, taking $\alpha = 1$. In (3) we obtain

$$(5) \quad \left| \tilde{B}_{n,1}(f, x) - \frac{f(x+) + f(x-)}{2} \right| \leq \frac{12(1+x) + x}{nx} \sum_{k=1}^n \omega_x(g_x, x/\sqrt{k}) \\ + \frac{8\sqrt{1+x}}{\sqrt{xn}} |f(x+) - f(x-)| + \frac{(2x)^\beta}{x^{2m}} O(n^{-[(m+1)/2]}).$$

It should be pointed out that estimation (5) is applicable to more general class of function and gives the better asymptotic estimate order than the main result of Wang and Guo ([2]) and the main result of Gupta and Kumar ([3]).

2. Auxiliary results

We need some auxiliary results for proving Theorem.

Lemma 1. *Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with the same geometric distribution*

$$P(\xi_1 = k) = \left(\frac{x}{1+x}\right)^k \frac{1}{1+x} \quad (k \in \mathbf{N}, x > 0 \text{ is a parameter}).$$

Then

$$(6) \quad E\xi_1 = x, E(\xi_1 - E\xi_1)^2 = x^2 + x, \text{ and } E|\xi_1 - E\xi_1|^3 \leq 3x(1+x)^2.$$

Proof. Direct calculation gives

$$\begin{aligned} E\xi_1 &= \sum_{k=0}^{\infty} k \left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = x, \\ E\xi_1^2 &= \sum_{k=0}^{\infty} k^2 \left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = 2x^2 + x, \\ E\xi_1^3 &= \sum_{k=0}^{\infty} k^3 \left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = 6x^3 + 6x^2 + x, \\ E\xi_1^4 &= \sum_{k=0}^{\infty} k^4 \left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = 24x^4 + 36x^3 + 14x^2 + x. \end{aligned}$$

Hence

$$E|\xi_1 - E\xi_1|^2 = x^2 + x, \text{ and } E(\xi_1 - E\xi_1)^4 = 9x^4 + 18x^3 + 10x^2 + x.$$

By Hölder inequality we get

$$\begin{aligned} E|\xi_1 - E\xi_1|^3 &\leq \sqrt{E(\xi_1 - E\xi_1)^4 E(\xi_1 - E\xi_1)^2} \leq \sqrt{(x^2 + x)(9x^4 + 18x^3 + 10x^2 + x)} \\ &\leq 3x(1+x)^2. \end{aligned}$$

The proof of Lemma 1 is complete. \square

The following Lemma 2 is the well-known Berry-Esseen bound for the central limit theorem of probability theory. It can be used to estimate upper bounds for the partial sum of Baskakov basis functions. Its proof can be found in Feller ([6], p.542) and Shiriyayev ([7], p.342).

Lemma 2. *Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of independent and identically distributed random variables with the expectation $E(\xi_1) = a_1$, the variance $E(\xi_1 - a_1)^2 = \sigma^2 >$*

0, $E|\xi_1 - a_1|^3 < \infty$, and let F_n stand for the distribution function of $\sum_{k=1}^n (\xi_k - a_1)/\sigma\sqrt{n}$. Then there exists a absolute constant $C, 1/\sqrt{2\pi} \leq C < 0.8$, such that for all t and n

$$(7) \quad \left| F_n(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2} du \right| \leq C \frac{E|\xi_1 - a_1|^3}{\sigma^3 \sqrt{n}}.$$

Lemma 3. For all $k \in \mathbf{N}$ and $x > 0$ there holds

$$(8) \quad b_{nk}(x) < \sqrt{\frac{1+x}{2ex}} \frac{1}{\sqrt{n}}.$$

Proof. In Theorem 2 of [8] we gave the optimal upper bound of Meyer-König and Zeller basis functions as

$$(9) \quad \binom{n+k-1}{k} t^k (1-t)^n < \frac{1}{\sqrt{2et}} \frac{1}{\sqrt{n}} \quad (t \in (0, 1]).$$

Replacing variable t by $\frac{x}{1+x}$ in the inequality (9) we get the optimal upper bound estimate:

$$b_{nk}(x) < \sqrt{\frac{1+x}{2ex}} \frac{1}{\sqrt{n}}. \quad \square$$

We need to estimate Lupas-Bézier basis functions $J_{nk}(x)$ and $J_{n-1,k}(x)$.

Lemma 4. For all $x \in (0, +\infty)$ and $k \in \mathbf{N}$ we have

$$(10) \quad |J_{nk}^\alpha(x) - J_{n-1,k+1}^\alpha(x)| \leq \frac{8\alpha\sqrt{1+1/x}}{\sqrt{n}},$$

and

$$(11) \quad |J_{nk}^\alpha(x) - J_{n-1,k}^\alpha(x)| \leq \frac{8\alpha\sqrt{1+1/x}}{\sqrt{n}}.$$

Proof. Since

$$|J_{n0}^\alpha(x) - J_{n-1,1}^\alpha(x)| \leq \alpha |J_{n0}(x) - J_{n-1,1}(x)| = \alpha b_{n-1,0}(x),$$

hence (10) holds for $k = 0$ form Lemma 3.

For $k = 1, 2, 3, \dots$, let $\{\xi_i\}_{i=1}^\infty$ be a sequence of independent random variables with the same geometric distribution

$$P(\xi_i = k) = \left(\frac{x}{1+x} \right)^k \frac{1}{1+x} \quad (k \in \mathbf{N}, x > 0 \text{ is a parameter}).$$

Let $\eta_n = \sum_{i=1}^n \xi_i$. Then the probability distribution of the random variable η_n is

$$P(\eta_n = k) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} = b_{nk}(x).$$

Set $A_1 = \frac{k-1-nx}{\sqrt{x(x+1)\sqrt{n}}}$, $A_2 = \frac{k-(n-1)x}{\sqrt{x(x+1)\sqrt{n-1}}}$, then

$$\begin{aligned} (12) \quad & |J_{nk}(x) - J_{n-1,k+1}(x)| = |P(\eta_n \geq k) - P(\eta_{n-1} \geq k+1)| \\ & = |1 - P(\eta_n \leq k-1) + P(\eta_{n-1} \leq k)| = |P(\eta_n \leq k-1) - P(\eta_{n-1} \leq k)| \\ & \leq \left| P(\eta_n \leq k-1) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_1} e^{-t^2/2} dt \right| + \left| P(\eta_{n-1} \leq k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_2} e^{-t^2/2} dt \right| \\ & \quad + \left| \frac{1}{\sqrt{2\pi}} \int_{A_1}^{A_2} e^{-t^2/2} dt \right| \end{aligned}$$

and using Lemma 2 and (6), we get

$$(13) \quad \left| P(\eta_n \leq k-1) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_1} e^{-t^2/2} dt \right| \leq C \frac{E|\xi_1 - a_1|^3}{\sqrt{n}\sigma_1^3} \leq \frac{2.4\sqrt{1+1/x}}{\sqrt{n}}$$

and

$$(14) \quad \left| P(\eta_{n-1} \leq k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_2} e^{-t^2/2} dt \right| \leq \frac{2.4\sqrt{1+1/x}}{\sqrt{n-1}} \leq \frac{3.6\sqrt{1+1/x}}{\sqrt{n}}.$$

Below we prove that

$$(15) \quad \left| \frac{1}{\sqrt{2\pi}} \int_{A_1}^{A_2} e^{-t^2/2} dt \right| \leq \frac{2\sqrt{1+1/x}}{\sqrt{n}}.$$

Direct calculation gives

$$\begin{aligned} (16) \quad & 0 \leq A_2 - A_1 \\ & = \frac{1}{\sqrt{x(x+1)n}} + \frac{x}{\sqrt{x(x+1)}(\sqrt{n} + \sqrt{n-1})} + \frac{k}{\sqrt{x(x+1)\sqrt{n}\sqrt{n-1}}(\sqrt{n} + \sqrt{n-1})} \\ & \leq \frac{1+x}{\sqrt{x(x+1)n}} + \frac{k}{\sqrt{x(x+1)\sqrt{n}\sqrt{n-1}}(\sqrt{n} + \sqrt{n-1})}. \end{aligned}$$

Hence if $k \leq 3nx + 3$, then

$$\begin{aligned} \left| \int_{A_1}^{A_2} e^{-t^2/2} dt \right| & \leq |A_2 - A_1| \leq \frac{1+x}{\sqrt{x(x+1)n}} + \frac{3nx+3}{\sqrt{x(x+1)\sqrt{n}\sqrt{n-1}}(\sqrt{n} + \sqrt{n-1})} \\ & \leq \frac{1+x}{\sqrt{x(x+1)\sqrt{n}}} + \frac{3x+3}{\sqrt{x(x+1)\sqrt{n}}} = \frac{4\sqrt{x+1}}{\sqrt{x}\sqrt{n}}. \end{aligned}$$

If $k > 3nx + 3$, then $A_1 > 0$ and

$$(17) \quad \frac{k}{A_1^2/2} = \frac{2nx(1+x)}{k-2nx-2+(nx+1)^2/k} < \frac{2nx(1+x)}{3nx+3-2nx-2} < 2(1+x).$$

From (16), (17) we have

$$\begin{aligned} \left| \int_{A_1}^{A_2} e^{-t^2/2} dt \right| &\leq e^{-A_1^2/2}(A_2 - A_1) \leq \frac{A_2 - A_1}{1 + A_1^2/2} \\ &\leq \frac{(1+x)/(\sqrt{x(x+1)n}) + k/(\sqrt{x(1+x)}\sqrt{n}\sqrt{n-1}(\sqrt{n} + \sqrt{n-1}))}{1 + A_1^2/2} \\ &\leq \frac{1+x}{\sqrt{x(x+1)}\sqrt{n}} + \frac{k/(\sqrt{x(x+1)}\sqrt{n})}{A_1^2/2} \\ &\leq \frac{1+x}{\sqrt{x(x+1)}\sqrt{n}} + \frac{2(1+x)}{\sqrt{x(x+1)}\sqrt{n}} \leq \frac{4\sqrt{x+1}}{\sqrt{x}\sqrt{n}}. \end{aligned}$$

Inequality (15) is proved.

From (12)-(15) and Lemma 3 we get

$$|J_{nk}(x) - J_{n-1,k+1}(x)| \leq \frac{8\sqrt{1+1/x}}{\sqrt{n}}.$$

Note that $J_{nk}(x) \leq 1$, $J_{n-1,k+1}(x) \leq 1$ and $\alpha \geq 1$, we deduce that

$$|J_{nk}^\alpha(x) - J_{n-1,k+1}^\alpha(x)| \leq \frac{8\alpha\sqrt{1+1/x}}{\sqrt{n}}.$$

The inequality (11) can be obtained by the similar method. □

Lemma 5. ([9], Lemma 1) *Let*

$$T_{n,m}(x) = (n-1) \sum_{k=0}^{\infty} b_{nk}(x) \int_0^{\infty} (t-x)^m b_{nk}(t) dt,$$

then

$$T_{n,0}(x) = 1, \quad T_{n,1}(x) = \frac{1+2x}{n-2}, \quad T_{n,2}(x) = \frac{2(n-1)x(1+x) + 2(1+2x)^2}{(n-2)(n-3)}$$

and

$$T_{n,m}(x) = O(n^{-[(m+1)/2]}).$$

Lemma 6. Let $x \in (0, \infty), 0 \leq y < x$. Then, for n sufficiently large we have

$$(18) \quad (n - 1) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \int_0^y b_{nk}(t) dt \leq \frac{4\alpha x(1+x)}{n(x-y)^2}.$$

Proof. Note that $Q_{nk}^{(\alpha)}(x) \leq \alpha b_{nk}(x), 0 \leq y < x$, by Lemma 5, we have for n sufficiently large

$$(19) \quad \begin{aligned} & (n - 1) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \int_0^y b_{nk}(t) dt \\ & \leq \alpha(n - 1) \sum_{k=0}^{\infty} b_{nk}(x) \int_0^y \left(\frac{x-t}{x-y}\right)^2 b_{nk}(t) dt \\ & \leq \frac{\alpha}{(x-y)^2} T_{n,2}(x) \leq \frac{4\alpha x(1+x)}{n(x-y)^2}. \end{aligned}$$

□

Lemma 7. Let m be an integer and $m \geq \beta/2, t \geq 2x$. Then

$$(20) \quad (n - 1) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \int_{2x}^{\infty} t^{\beta} b_{nk}(t) dt \leq \frac{\alpha(2x)^{\beta}}{x^{2m}} O(n^{-[(m+1)/2]}).$$

Proof. For $m \geq \beta/2$ and $t \geq 2x, \frac{t^{\beta}}{(t-x)^{2m}}$ is monotone decreasing for variable t . Note that $Q_{nk}^{(\alpha)}(x) \leq \alpha b_{nk}(x)$. Hence by Lemma 6, we get

$$\begin{aligned} & (n - 1) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \int_{2x}^{\infty} t^{\beta} b_{nk}(t) dt \\ & \leq \frac{\alpha(2x)^{\beta}}{x^{2m}} (n - 1) \sum_{k=0}^{\infty} b_{nk}(x) \int_{2x}^{\infty} (t-x)^{2m} b_{nk}(t) dt \\ & \leq \frac{\alpha(2x)^{\beta}}{x^{2m}} T_{n,m}(x) = \frac{\alpha(2x)^{\beta}}{x^{2m}} O(n^{-[(m+1)/2]}). \end{aligned}$$

□

3. Proof of the Theorem

Let $f \in \Phi_{loc,\beta}$ and $f(x+), f(x-)$ exist at x , then $f(t)$ can be expressed as

$$(21) \quad \begin{aligned} f(t) = & \frac{f(x+) + \alpha f(x-)}{\alpha + 1} + g_x(t) + \frac{f(x+) - f(x-)}{\alpha + 1} \operatorname{sgn}_{\alpha}((t-x), x) \\ & + \delta_x(t) \left[f(x) - \frac{f(x+) + \alpha f(x-)}{\alpha + 1} \right] \end{aligned}$$

where $g_x(t)$ is defined as (4), and

$$\operatorname{sgn}_\alpha(t) = \begin{cases} \alpha, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}; \quad \delta_x(t) = \begin{cases} 1, & t = x \\ 0, & t \neq x. \end{cases}$$

In fact, if $t > x$, then

$$f(t) = \frac{f(x+) + \alpha f(x-)}{\alpha + 1} + f(t) - f(x+) + \frac{f(x+) - f(x-)}{\alpha + 1}\alpha,$$

if $t < x$, then

$$f(t) = \frac{f(x+) + \alpha f(x-)}{\alpha + 1} + f(t) - f(x-) - \frac{f(x+) - f(x-)}{\alpha + 1}.$$

Hence (21) holds. Obviously, $\tilde{B}_{n,\alpha}(\delta_x, x) = 0$, hence we have

$$\begin{aligned} (22) \quad & \left| \tilde{B}_{n,\alpha}(f, x) - \frac{f(x+) - \alpha f(x-)}{\alpha + 1} \right| \\ & \leq |\tilde{B}_{n,\alpha}(g_x, x)| + \frac{|f(x+) - f(x-)|}{\alpha + 1} |\tilde{B}_{n,\alpha}(\operatorname{sgn}_\alpha(t - x), x)|. \end{aligned}$$

We first estimate $|\tilde{B}_{n,\alpha}(\operatorname{sgn}_\alpha(t - x), x)|$. By direct calculation

$$\begin{aligned} (23) \quad & \tilde{B}_{n,\alpha}(\operatorname{sgn}_\alpha(t - x), x) \\ & = (n - 1) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \left(\int_x^{\infty} \alpha b_{nk}(t) dt - \int_0^x b_{nk}(t) dt \right) \\ & = (n - 1) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \left(\int_0^{\infty} \alpha b_{nk}(t) dt - (1 + \alpha) \int_0^x b_{nk}(t) dt \right) \\ & = \alpha - (1 + \alpha)(n - 1) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \int_0^x b_{nk}(t) dt. \end{aligned}$$

Since

$$J'_{n-1,k+1}(x) = \sum_{j=k+1}^{\infty} b'_{n-1,j}(x) = (n - 1) \sum_{j=k+1}^{\infty} (b_{n,j-1}(x) - b_{n,j}(x)) = (n - 1)b_{nk}(x),$$

and $J_{n-1,k+1}(0) = 0$, so

$$(24) \quad (n - 1) \int_0^x b_{nk}(t) dt = J_{n-1,k+1}(x) = 1 - \sum_{j=0}^k b_{n-1,j}(x).$$

From (23), (24)

$$\begin{aligned} \tilde{B}_{n,\alpha}(\operatorname{sgn}_\alpha(t-x), x) &= \alpha - (1 + \alpha) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \left(1 - \sum_{j=0}^k b_{n-1,j}(x) \right) \\ &= (1 + \alpha) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \sum_{j=0}^k b_{n-1,j}(x) - 1. \end{aligned}$$

Noticing that $\sum_{k=0}^{\infty} \sum_{j=0}^k \bullet = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \bullet$, we have

$$\begin{aligned} (25) \quad & (1 + \alpha) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \sum_{j=0}^k b_{n-1,j}(x) - 1 \\ &= (1 + \alpha) \sum_{j=0}^{\infty} b_{n-1,j}(x) \sum_{k=j}^{\infty} Q_{nk}^{(\alpha)}(x) - 1 \\ &= (1 + \alpha) \sum_{j=0}^{\infty} b_{n-1,j}(x) J_{nj}^\alpha(x) - 1 \\ &= (1 + \alpha) \sum_{j=0}^{\infty} b_{n-1,j}(x) J_{nj}^\alpha(x) - \sum_{j=0}^{\infty} Q_{n-1,j}^{(\alpha+1)}(x). \end{aligned}$$

By mean value theorem

$$Q_{n-1,j}^{(\alpha+1)}(x) = J_{n-1,j}^{\alpha+1}(x) - J_{n-1,j+1}^{\alpha+1}(x) = (\alpha + 1)b_{n-1,j}(x)\gamma_{nj}^\alpha(x)$$

where $J_{n-1,j+1}(x) < \gamma_{nj}(x) < J_{n-1,j}(x)$.

Hence it follows from Lemma 4 and (25) that

$$\begin{aligned} (26) \quad |\tilde{B}_{n,\alpha}(\operatorname{sgn}_\alpha(t-x), x)| &= \left| (1 + \alpha) \sum_{j=0}^{\infty} b_{n-1,j}(x) (J_{nj}^\alpha(x) - \gamma_{nj}^\alpha(x)) \right| \\ &\leq (1 + \alpha) \sum_{j=0}^{\infty} b_{n-1,j}(x) \frac{8\alpha\sqrt{1+x}}{\sqrt{nx}} \\ &\leq \frac{8\alpha(1 + \alpha)\sqrt{1+x}}{\sqrt{nx}}. \end{aligned}$$

Next, we estimate $|\tilde{B}_{n,\alpha}(g_x, x)|$. Let

$$K_{n,\alpha}(x, t) = \int_0^t (n-1) \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) b_{nk}(u) du.$$

We recall the Lebesgue-Stieltjes integral representations:

$$(27) \quad \tilde{B}_{n,\alpha}(g_x, x) = \int_0^\infty g_x(t) d_t K_{n,\alpha}(x, t).$$

We decompose the integral of (27) into four parts, as

$$\int_0^\infty g_x(x) d_t K_{n,\alpha}(x, t) = \Delta_{1,n} + \Delta_{2,n} + \Delta_{3,n} + \Delta_{4,n}$$

where

$$\begin{aligned} \Delta_{1,n} &= \int_0^{x-x/\sqrt{n}} g_x(t) d_t K_{n,\alpha}(x, t), & \Delta_{2,n} &= \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} g_x(t) d_t K_{n,\alpha}(x, t) \\ \Delta_{3,n} &= \int_{x+x/\sqrt{n}}^{2x} g_x(t) d_t K_{n,\alpha}(x, t), & \Delta_{4,n} &= \int_{2x}^\infty g_x(t) d_t K_{n,\alpha}(x, t). \end{aligned}$$

We shall evaluate $\Delta_{1,n}, \Delta_{2,n}, \Delta_{3,n}$ and $\Delta_{4,n}$ with the quantity $\omega_x(g_x, \lambda)$. First, note that $g_x(x) = 0$ we have

$$(28) \quad \begin{aligned} |\Delta_{2,n}| &\leq \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} |g_x(t) - g_x(x)| d_t K_{n,\alpha}(x, t) \\ &\leq \omega_x(g_x, x/\sqrt{n}) \leq \frac{1}{n} \sum_{k=1}^n \omega_x(g_x, x/\sqrt{k}). \end{aligned}$$

Next we estimate $|\Delta_{1,n}|$. Note that $\omega_x(g_x, \lambda)$ is monotone non-decreasing with respect to λ , it follows that

$$|\Delta_{1,n}| = \left| \int_0^{x-x/\sqrt{n}} g_x(t) d_t K_{n,\alpha}(x, t) \right| \leq \int_0^{x-x/\sqrt{n}} \omega_x(g_x, x-t) d_t K_{n,\alpha}(x, t).$$

Using partial integration with $y = x - x/\sqrt{n}$, we have

$$(29) \quad \begin{aligned} &\int_0^{x-x/\sqrt{n}} \omega_x(g_x, x-t) d_t K_{n,\alpha}(x, t) \\ &\leq \omega_x(g_x, x-y) K_{n,\alpha}(x, y+) + \int_0^y \hat{K}_{n,\alpha}(x, t) d(-\omega_x(g_x, x-t)) \end{aligned}$$

where $\hat{K}_{n,\alpha}(x, t)$ is the normalized form of $K_{n,\alpha}(x, t)$. Since $\hat{K}_{n,\alpha}(x, t) \leq K_{n,\alpha}(x, t)$ on $(0, \infty)$, from (29) it follows that

$$(30) \quad |\Delta_{1,n}| \leq \omega_x(g_x, x-y) \frac{4\alpha x(1+x)}{n(x-y)^2} + \frac{4\alpha x(1+x)}{n} \int_0^y \frac{1}{(x-t)^2} d(-\omega_x(g_x, x-t)).$$

Since

$$\begin{aligned} & \int_0^y \frac{1}{(x-t)^2} d(-\omega_x(g_x, x-t)) \\ &= -\frac{1}{(x-t)^2} \omega_x(g_x, x-t) \Big|_0^{y^+} + \int_0^y \omega_x(g_x, x-t) \frac{2}{(x-t)^3} dt \\ &= -\frac{\omega_x(g_x, x-y)}{(x-y)^2} + \frac{\omega_x(g_x, x)}{x^2} + \int_0^y \omega_x(g_x, x-t) \frac{2}{(x-t)^3} dt. \end{aligned}$$

So we have from (30)

$$|\Delta_{1,n}| \leq \frac{4\alpha x(1+x)}{nx^2} \omega_x(g_x, x) + \frac{4\alpha x(1+x)}{n} \int_0^{x-x/\sqrt{n}} \omega_x(g_x, x-t) \frac{2}{(x-t)^3} dt.$$

Putting $t = x - x/\sqrt{u}$ for the last integral we get

$$\int_0^{x-x/\sqrt{n}} \omega_x(g_x, x-t) \frac{2}{(x-t)^3} dt = \frac{1}{x^2} \int_1^n \omega_x(g_x, x/\sqrt{u}) du \leq \frac{1}{x^2} \sum_{k=1}^n \omega_x(g_x, x/\sqrt{k}).$$

Consequently

$$\begin{aligned} (31) \quad |\Delta_{1,n}| &\leq \frac{4\alpha x(1+x)}{nx} \left(\omega_x(g_x, x) + \sum_{k=1}^n \omega_x(g_x, x/\sqrt{k}) \right) \\ &\leq \frac{8\alpha(1+x)}{nx} \sum_{k=1}^n \omega_x(g_x, x/\sqrt{k}). \end{aligned}$$

Using the similar method to estimate $|\Delta_{3,n}|$ we get

$$(32) \quad |\Delta_{3,n}| \leq \frac{8\alpha(1+x)}{nx} \sum_{k=1}^n \omega_x(g_x, x/\sqrt{k}).$$

Finally, by assumption $g_x(t) = O(t^\beta)$ for some $\beta > 0$ as $t \rightarrow \infty$, using Lemma 7, we have

$$\begin{aligned} (33) \quad |\Delta_{4,n}| &= \left| \int_{2x}^\infty g_x(t) d_t K_{n,\alpha}(x, t) \right| \\ &\leq M(n-1) \sum_{k=0}^\infty Q_{nk}^{(\alpha)}(x) \int_{2x}^\infty t^\beta b_{nk}(t) dt = \frac{\alpha(2x)^\beta}{x^{2m}} O(n^{-[(m+1)/2]}) \end{aligned}$$

where M is a positive constant.

From (28), (31), (32) and (33), we get

$$\begin{aligned} (34) \quad |\tilde{B}_{n,\alpha}(g_x, x)| &\leq |\Delta_{1,n}| + |\Delta_{2,n}| + |\Delta_{3,n}| + |\Delta_{4,n}| \\ &\leq \frac{16\alpha(1+x) + x}{nx} \sum_{k=1}^n \omega_x(g_x, x/\sqrt{k}) + \frac{\alpha(2x)^\beta}{x^{2m}} O(n^{-[(m+1)/2]}). \end{aligned}$$

The inequality (3) now follows from (22), (26) and (34). The proof of Theorem is complete. \square

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