KYUNGPOOK Math. J. 43(2003), 579-585

On a Class of Univalent Functions Defined by Ruscheweyh Derivatives

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Abstract. A new class of univalent functions is defined by making use of the Ruscheweyh derivatives. We provide necessary and sufficient coefficient conditions, extreme points, integral representations, distortion bounds, and radius of starlikeness and convexity for this class.

1. Introduction

Let $\mathcal{A}$ denote the family of functions f that are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$ and consider the subclass $\mathcal{T}$ consisting of functions f in $\mathcal{A}$, which are univalent in Δ and are of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, where $a_n \geq 0$. For $\alpha \geq 0,\ 0 \leq \beta < 1$ and $\lambda > -1$, we let $\mathcal{D}(\alpha, \beta, \lambda)$ consist of functions f in $\mathcal{T}$ satisfying the condition

$$\Re\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}\right) > \alpha\left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right| + \beta. \tag{1}$$

The operator $D^\lambda f$ is the Ruscheweyh derivative [2] of f defined by

$$D^\lambda f(z) = \frac{z(z^{\lambda-1}f(z))^{(\lambda)}}{\lambda!} = \frac{z}{(1-z)^{\lambda+1}} * f(z) = z - \sum_{n=2}^{\infty} B_n(\lambda) a_n z^n$$

where

$$B_n(\lambda) = \binom{n+\lambda-1}{\lambda} = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}{(n-1)!}.$$

Here the operation $*$ stands for the convolution of two power series $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ denoted by $(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$.

Received November 7, 2002, and, in revised form, February 7, 2003.
2000 Mathematics Subject Classification: Primary 30C45, Secondary 30C50.
Key words and phrases: analytic, univalent, convex, starlike, ruscheweyh derivative.

The family $\mathcal{D}(\alpha, \beta, \lambda)$ is of special interest for it contains many well-known as well as new classes of analytic univalent functions. In particular, for $\alpha = 0$ and $0 \leq \lambda \leq 1$ it provides a transition from starlike functions to convex functions. More specifically, $\mathcal{D}(0, \beta, 0)$ is the family of functions starlike of order β and $\mathcal{D}(0, \beta, 1)$ is the family of functions convex of order β. For $\mathcal{D}(\alpha, 0, 0)$, we obtain the class of uniformly α-starlike functions introduced by Kanas and Wisniowska ([1]), which can be generalized to, $\mathcal{D}(\alpha, \beta, 0)$, the class of uniformly α-starlike functions of order β. Generally speaking, $\mathcal{D}(\alpha, \beta, \lambda)$, consists of functions $F(z) = D^\lambda f(z)$ which are uniformly α-starlike of order β in Δ. In this paper we provide necessary and sufficient coefficient conditions, extreme points, integral representations, distortion bounds, and radius of starlikeness and convexity for functions in $\mathcal{D}(\alpha, \beta, \lambda)$.

2. Main results

First we provide a necessary and sufficient coefficient bound for functions in $\mathcal{D}(\alpha, \beta, \lambda)$.

Theorem 2.1. *Let $f \in \mathcal{T}$. Then f is in $\mathcal{D}(\alpha, \beta, \lambda)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{n(1+\alpha) - (\alpha+\beta)}{1-\beta}\, a_n B_n(\lambda) < 1. \tag{2}$$

Proof. Suppose that $f \in \mathcal{D}(\alpha, \beta, \lambda)$. Using the fact that $\Re\, w > \alpha|w-1| + \beta$ if and only if $\Re(w(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}) > \beta$ for real γ and letting $w = z(D^\lambda f)'/D^\lambda f$ in (1) we obtain

$$\Re\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}\right) > \beta$$

or equivalently

$$\Re\left[\frac{(1-\beta) - \sum_{n=2}^{\infty}(n-\beta)a_n B_n(\lambda) z^{n-1} - \alpha e^{i\gamma}\sum_{n=2}^{\infty}(n-1)a_n B_n(\lambda) z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n B_n(\lambda) z^{n-1}}\right] > 0.$$

The above inequality must hold for all z in Δ. Letting $z \to 1^-$ yields

$$\Re\left[\frac{(1-\beta) - \sum_{n=2}^{\infty}(n-\beta)a_n B_n(\lambda) - \alpha e^{i\gamma}\sum_{n=2}^{\infty}(n-1)a_n B_n(\lambda)}{1 - \sum_{n=2}^{\infty} a_n B_n(\lambda)}\right] > 0$$

and so by the mean value theorem we have

$$Re\left[(1-\beta) - \sum_{n=2}^{\infty}(n-\beta)a_n B_n(\lambda) - \alpha e^{i\gamma}\sum_{n=2}^{\infty}(n-1)a_n B_n(\lambda)\right] > 0.$$

Therefore

$$\sum_{n=2}^{\infty}(n(1+\alpha) - (\alpha+\beta))a_n B_n(\lambda) < 1-\beta.$$

Conversely, let (2) hold. We will show that (1) is satisfied and so $f \in \mathcal{D}(\alpha,\beta,\lambda)$. Using the fact that $\Re(w) > \alpha$ if and only if $|w-(1+\alpha)| < |w+(1-\alpha)|$ it is enough to show that

$$\begin{aligned}&\left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - \left(1+\alpha\left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right| + \beta\right)\right| \\ < \ &\left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} + \left(1-\alpha\left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right| - \beta\right)\right|.\end{aligned}$$

For letting $e^{i\phi} = \frac{D^\lambda f(z)}{|D^\lambda f(z)|}$ we may write

$$\begin{aligned}E &= \left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} + \left(1-\alpha\left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right| - \beta\right)\right| \\ &= \frac{1}{|D^\lambda f(z)|}\left|z(D^\lambda f(z))' + (1-\beta)D^\lambda f(z) - \alpha e^{i\phi}|z(D^\lambda f(z))' - D^\lambda f(z)|\right| \\ &> \frac{|z|}{|D^\lambda f(z)|}\left[(2-\beta) - \sum_{n=2}^{\infty}(n+1-\beta+n\alpha-\alpha)a_nB_n(\lambda)\right].\end{aligned}$$

and

$$\begin{aligned}F &= \left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - \left(1+\alpha\left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right| + \beta\right)\right| \\ &= \frac{1}{|D^\lambda f(z)|}\left|-\beta z - \sum_{n=2}^{\infty}(n-1-\beta)a_nB_n(\lambda)z^n - \alpha e^{i\phi}\left|\sum_{n=2}^{\infty}(1-n)a_nB_n(\lambda)z^n\right|\right| \\ &< \frac{|z|}{|D^\lambda f(z)|}\left[\beta + \sum_{n=2}^{\infty}(n-1-\beta+n\alpha-\alpha)a_nB_n(\lambda)\right].\end{aligned}$$

It is easy to verify that $E - F > 0$ if (2) holds and so the proof is complete. □

Remark 2.2. The above theorem for the special cases $\mathcal{D}(0,\beta,0)$ and $\mathcal{D}(0,\beta,1)$ lead to results obtained by Silverman ([3]).

Remark 2.3. Since $B_n(\lambda_2) < B_n(\lambda_1)$ for $\lambda_2 < \lambda_1$ we note that $\mathcal{D}(\alpha,\beta,\lambda_1) \subset \mathcal{D}(\alpha,\beta,\lambda_2)$.

The extreme points and integral representation for the class $\mathcal{D}(\alpha,\beta,\lambda)$ are given in the next two theorems.

Theorem 2.4. *Let $f_1(z) = z$ and $f_n(z) = z - \frac{1-\beta}{[n(1+\alpha)-(\alpha+\beta)]B_n(\lambda)}z^n$ where $n = 2,3,\cdots$. Then $f \in \mathcal{D}(\alpha,\beta,\lambda)$ if and only if it can be expressed in the form $f(z) = \sum_{n=2}^{\infty}\mu_n f_n(z)$ where $\mu_n \geq 0$ and $\sum_{n=1}^{\infty}\mu_n = 1$. In particular, the extreme points of $\mathcal{D}(\alpha,\beta,\lambda)$ are the functions $f_1(z) = z$ and*

$$f_n(z) = z - \frac{(1-\beta)}{[n(1+\alpha)-(\alpha+\beta)]B_n(\lambda)}z^n, \quad n = 2,3,\cdots.$$

Proof. First let f be expressed as in the above theorem. This means that we can write

$$f(z)=\sum_{n=1}^{\infty}\mu_n f_n(z)=z-\sum_{n=2}^{\infty}\frac{(1-\beta)\mu_n}{[n(1+\alpha)-(\alpha+\beta)]B_n(\lambda)}z^n=z-\sum_{n=2}^{\infty}t_n z^n.$$

Therefore $f\in\mathcal{D}(\alpha,\beta,\lambda)$ since

$$\sum_{n=2}^{\infty}\frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta}\quad t_nB_n(\lambda)=\sum_{n=2}^{\infty}\mu_n=1-\mu_1<1.$$

Conversely, suppose that $f\in\mathcal{D}(\alpha,\beta,\lambda)$. Then, by (2), we have

$$a_n<\frac{1-\beta}{[n(1+\alpha)-(\alpha+\beta)]B_n(\lambda)},\quad n=2,3,\cdots .$$

So, we may set

$$\mu_n=\frac{[n(1+\alpha)-(\alpha+\beta)]B_n(\lambda)}{1-\beta}\ a_n,\quad n=2,3,\cdots$$

and $\mu_1=1-\sum_{n=2}^{\infty}\mu_n$. Then

$$\begin{aligned}f(z)&=z-\sum_{n=2}^{\infty}a_nz^n=z-\sum_{n=2}^{\infty}\frac{1-\beta}{[n(1+\alpha)-(\alpha+\beta)]B_n(\lambda)}\mu_nz^n\\&=z-\sum_{n=2}^{\infty}\mu_n[z-f_n(z)]\\&=\left(1-\sum_{n=2}^{\infty}\mu_n\right)z-\sum_{n=2}^{\infty}\mu_nf_n(z)=\sum_{n=1}^{\infty}\mu_nf_n(z).\end{aligned}$$

This completes the proof. □

For $\alpha=\lambda=0$ and $\alpha=\lambda-1=0$ we obtain the following respective corollaries which have also been obtained by Silverman ([3]).

Corollary 2.5. *Let $f_1(z)=z$ and $f_n(z)=z-\frac{1-\beta}{n-\beta}z^n$, $n=2,3,\cdots$ then $f\in S^*(\beta)$ if and only if it can be expressed in the form $f(z)=\sum_{n=1}^{\infty}\mu_nf_n(z)$, $\mu_n\geq 0$, $\sum_{n=1}^{\infty}\mu_n=1$.*

Corollary 2.6. *Let $f_1(z)=z$ and $f_n(z)=z-\frac{1-\beta}{n(n-\beta)}z^n$, $n=2,3,\cdots$, then $f\in K(\beta)$ if and only if it can be expressed in the form $f(z)=\sum_{n=1}^{\infty}\mu_nf_n(z)$, $\mu_n\geq 0$, $\sum_{n=1}^{\infty}\mu_n=1$.*

The following theorem provides integral representations for $D^{\lambda}f$.

Theorem 2.7. *Let $f \in \mathcal{D}(\alpha, \beta, \lambda)$ then*

$$D^\lambda f(z) = exp\left(\int_0^z \frac{\alpha + \beta Q(t)}{t(\alpha - Q(t))} dt\right) \tag{3}$$

where $|Q(z)| < 1$. Also

$$D^\lambda f(z) = z \exp\left(\int_x \log(\alpha - xz)^{-(1+\beta)} d\mu(x)\right) \tag{4}$$

where $\mu(x)$ is probability measure on $X = \{x||x| = 1\}$.

Proof. The case $\alpha = 0$ is obvious. Let $\alpha \neq 0$. Then for $f \in \mathcal{D}(\alpha, \beta, \lambda)$ and $w = \frac{z(D^\lambda f(z))'}{D^\lambda f(z)}$ we have $\Re\, w > \alpha|w - 1| + \beta$. Therefore $\left|\frac{w-1}{w-\beta}\right| < \frac{1}{\alpha}$ and $\frac{w-1}{w-\beta} = \frac{Q(z)}{\alpha}$ where $|Q(z)| < 1$. This yields

$$\frac{(D^\lambda f(z))'}{D^\lambda f(z)} = \frac{\alpha - \beta Q(z)}{z(\alpha - Q(z))}$$

and therefore

$$D^\lambda f(z) = \exp\left(\int_0^z \frac{\alpha - \beta Q(t)}{t(\alpha - Q(t))} dt\right).$$

For the second representation, set $X = \{x : |x| = 1\}$. Then we have, $\frac{w-1}{w-\beta} = \frac{1}{\alpha} xz$ or

$$\frac{(D^\lambda f(z))'}{D^\lambda f(z)} = \frac{\alpha - \beta xz}{z(\alpha - xz)} \Rightarrow \log \frac{D^\lambda f(z)}{z} = -(1 + \beta) \log(\alpha - xz).$$

If $\mu(x)$ is the probability measure on X then

$$D^\lambda f(z) = z \exp\left(\int_X \log(\alpha - xz)^{-(1+\beta)} d\mu(x)\right).$$

□

Next we obtain a distortion bound for $D^\lambda f$.

Theorem 2.8. *Let $f \in \mathcal{D}(\alpha, \beta, \lambda)$, then*

$$|z| - \frac{1 - \beta}{2 - \alpha - \beta}|z|^2 < |D^\lambda f(z)| < |z| + \frac{1 - \beta}{2 - \alpha - \beta}|z|^2. \tag{5}$$

Proof. For $f \in \mathcal{D}(\alpha, \beta, \lambda)$ we have $\sum_{n=2}^\infty a_n B_n(\lambda) < \frac{1-\beta}{2-\alpha-\beta}$. Therefore

$$|D^\lambda f(z)| \leq |z| + \sum_{n=2}^\infty a_n B_n(\lambda)|z|^n \leq |z| + |z|^2 \sum_{n=2}^\infty a_n B_n(\lambda) < |z| + \frac{1 - \beta}{2 - \alpha - \beta}|z|^2,$$

and

$$|D^\lambda f(z)| \geq |z| - \sum_{n=2}^{\infty} a_n B_n(\lambda)|z|^n \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n B_n(\lambda) > |z| - \frac{1-\beta}{2-\alpha-\beta}|z|^2.$$

□

Finally, we obtain the radius of starlikeness and convexity.

Theorem 2.9. *Let $f \in \mathcal{D}(\alpha, \beta, \lambda)$. Then $f(z)$ is starlike of order μ $(0 \leq \mu < 1)$ in $|z| < r(\mu, \alpha, \beta, \lambda)$ where*

$$r(\mu, \alpha, \beta, \lambda) = \inf_n \left[\frac{[n(1+\alpha) - (\alpha+\beta)](1-\mu)}{(1-\beta)(n-\mu)} B_n(\lambda) \right]^{\frac{1}{n-1}}. \tag{6}$$

Proof. For $0 \leq \mu < 1$ we need to show that $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \mu$. In other words, it is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n|z|^{n-1}} \; < \; 1 - \mu$$

or

$$\sum_{n=2}^{\infty} \frac{n-\mu}{1-\mu} a_n |z|^{n-1} < 1. \tag{7}$$

It is easy to see that (7) holds if

$$|z|^{n-1} < \frac{[n(1+\alpha) - (\alpha+\beta)](1-\mu)}{(1-\beta)(n-\mu)} B_n(\lambda).$$

This completes the proof. □

Upon noting the fact that f is convex if and only if zf' is starlike, we obtain

Theorem 2.10. *Let $f \in \mathcal{D}(\alpha, \beta, \lambda)$. Then $f(z)$ is convex of order μ $(0 \leq \mu < 1)$ in $|z| < r(\mu, \alpha, \beta, \lambda)$ where*

$$r(\mu, \alpha, \beta, \lambda) = \inf_n \left[\frac{[n(1+\alpha) - (\alpha+\beta)](1-\mu)}{n(1-\mu)(n-\mu)} B_n(\lambda) \right]^{\frac{1}{n-1}}.$$

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