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Strongly Unique Best Coapproximation

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Abstract. This paper delineates some fundamental properties of the set of strongly unique best coapproximation. Uniqueness of strongly unique best coapproximation is studied. Some characterizations of strongly unique best coapproximation and strongly unique best approximation are obtained. Some more results concerning strongly unique best uniform coapproximation and strongly unique best uniform approximation are presented. Some relations between best uniform approximation and strongly unique best uniform coapproximation are established.

1. Introduction

A new kind of approximation was first introduced in 1979 by Franchetti and Furi ([2]) to characterize real Hilbert spaces among real reflexive Banach spaces. This was christened 'best coapproximation' by Papini and Singer ([15]). Subsequently, Geetha S. Rao and coworkers have developed this theory to a considerable extent ([3], [4], [5], [6], [7], [8], [9], [10], [11], [12]). This theory is largely concerned with the questions of existence, uniqueness and characterizations of best coapproximation. It also deals with the continuity properties of the cometric projection and selections for the cometric projection, apart from related maps and strongly unique best coapproximation. This paper mainly deals with some characterizations of strongly unique best coapproximation with respect to L_∞-norm. Section 2 gives some fundamental concepts of best approximation and best coapproximation. Section 3 delineates some fundamental results related to strongly unique best coapproximation. It is observed that a strongly unique best coapproximation is not unique in general and it is proved that it is unique in an inner product space. Section 4 establishes some necessary and sufficient conditions characterizing strongly unique best coapproximation and strongly unique best approximation. Section 5

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provides some results concerning strongly unique best uniform coapproximation. Under which conditions a function belongs to the closure of the set of functions which have strongly unique best uniform coapproximation from a subset is investigated. Section 6 discusses some relations between best uniform approximation and strongly unique best uniform coapproximation.

2. Preliminaries

Definition 2.1. Let G be a nonempty subset of a real normed linear space X. An element $g_f \in G$ is called a *best coapproximation* to $f \in X$ from G if for every $g \in G$,

$$\|g - g_f\| \leq \|f - g\|.$$

The set of all best coapproximations to $f \in X$ from G is denoted by $R_G(f)$. The subset G is called an *existence set* if $R_G(f)$ contains at least one element, for every $f \in X$. The subset G is called a *uniqueness set* if $R_G(f)$ contains at most one element, for every $f \in X$. The subset G is called an *existence and uniqueness set* if $R_G(f)$ contains exactly one element, for every $f \in X$.

Definition 2.2. Let G be a nonempty subset of a real normed linear space X. The set-valued mapping $R_G : X \to \mathrm{POW}(G)$ which associates for every $f \in X$, the set $R_G(f)$ of the best coapproximations to f from G is called the *cometric projection* onto G, where POW(G) denotes the set of all subsets of G.

Definition 2.3. Let G be a nonempty subset of a real normed linear space X. An element $g_f \in G$ is called a *best approximation* to $f \in X$ from G if for every $g \in G$, $\|f - g_f\| \leq \|f - g\|$ i.e., if $\|f - g_f\| = \inf_{g \in G} \|f - g\| = d(f, G)$,
where $d(f, G)$:= distance between the element f and the set G.

The set of all best approximations to $f \in X$ from G is denoted by $P_G(f)$. The subset G is called a *proximinal* or *existence set* if $P_G(f)$ contains at least one element for every $f \in X$. G is called a *semi Chebyshev* or *uniqueness set* if $P_G(f)$ contains at most one element for every $f \in X$. G is called a *Chebyshev* or *existence* and *uniqueness set* if $P_G(f)$ contains exactly one element for every $f \in X$.

Let $[a, b]$ be a closed and bounded interval on the real line. A space of continuous real valued functions on $[a, b]$ is defined by

$$C[a, b] = \{f : [a, b] \to \mathcal{R} : f \quad \text{is} \quad \text{continuous}\},$$

where $\mathcal{R}$ denotes the set of real numbers.

Definition 2.4. For all functions $f \in C[a, b]$, the *uniform norm* or L_∞*-norm* or *supremum norm* is defined by $\|f\|_\infty = \sup_{t \in [a,b]} |f(t)|$.

Best coapproximation (respectively, best approximation) with respect to this norm is called *best uniform coapproximation* (respectively, *best uniform approximation*).

Definition 2.5. Let G be a nonempty subset of a real normed linear space X. The set-valued mapping $P_G : X \to \mathrm{POW}(G)$ which associates for every $f \in X$, the set $P_G(f)$ of the best approximations to f from G is called the *metric projection* onto G.

Definition 2.6. Let G be a linear subspace of a real normed linear space X and let $f_1, f_2 \in X$. Then f_1 is *orthogonal* to f_2 (denoted by $f_1 \perp f_2$) if $\|f_1\| \leq \|f_1+\alpha f_2\|$ for every $\alpha \in \mathcal{R}$. The element $f \in X$ is said to be orthogonal to the subset G (denoted by $f \perp G$) iff $f \perp g$ for every $g \in G$. Similarly, $G \perp f$ if $g \perp f$ for every $g \in G$.

Let G be a linear subspace of a real normed linear space X, $f \in X$ and $g_f \in G$. It is clear from the definitions of best approximation, best coapproximation and the above notion of orthogonality that $g_f \in R_G(f)$ if and only if $G \perp (f - g_f)$ and $g_f \in P_G(f)$ if and only if $(f - g_f) \perp G$. This notion of orthogonality is not symmetric in an arbitrary normed linear space. But this orthogonality is symmetric in an inner product space. Hence best approximation and best coapproximation coincide in an inner product space. A detailed discussion of this can be found in [2], [15].

For sake of brevity, the terminology subspace is used instead of a linear subspace. Unless otherwise stated all normed linear spaces considered in this paper are real normed linear spaces.

3. Some fundamental results

Definition 3.1. Let G be a subset of a normed linear space X, $f \in X \setminus G$ and $g_f \in G$. Then g_f is called a *strongly unique best approximation* to f from G, if there exists a constant $k_f > 0$ such that for all $g \in G$,

$$\|f - g_f\| \;\leq\; \|f - g\| - k_f\|g - g_f\|. \tag{1}$$

Similarly, g_f is called a *strongly unique best coapproximation* to f from G, if there exists a constant $k_f > 0$ such that for all $g \in G$,

$$\|g - g_f\| \;\leq\; \|f - g\| - k_f\|f - g_f\|. \tag{2}$$

It is clear that if some $k_f > 0$ satisfies inequality (1) (respectively, (2)), then every smaller value of k_f will also satisfy inequality (1) (respectively, (2)). The maximum of all such numbers $k_f > 0$ is called the strong unicity constant of f and is denoted by $K(f)$.

Obviously, every strongly unique best approximation is a unique best approximation. But strongly unique best coapproximation need not imply the uniqueness, which requires further investigation. It is clear that g_f is a strongly unique best coapproximation to f from G with the corresponding strong unicity constant equal to 1 if and only if g_f is a strongly unique best approximation to f from G with the strong unicity constant equal to 1. Thus the strongly unique best coapproximation implies the uniqueness if the corresponding strong unicity constant is equal to 1.

It is clear that the strong unicity constant in the context of coapproximation is bounded by 1, for if g_f is a strongly unique best coapproximation to f from G, then for all $g \in G$,

$$\begin{aligned} k_f\|f-g_f\| &\leq \|f-g\|-\|g-g_f\| \\ &\leq \|f-g_f\|. \end{aligned}$$

Hence $k_f \leq 1$. This implies that $K(f) \leq 1$.

It is not possible to say that $\|f-g_f\|$ is small whenever $\|f-g\|-\|g-g_f\|$ is small, for $g \in G$. However, it is true if $g_f \in G$ is a strongly unique best coapproximation to f from G, since then for all $g \in G$,

$$\|f-g_f\| \leq \frac{1}{k_f}(\|f-g\|-\|g-g_f\|).$$

If $f \in G$, then all values of $k_f > 0$ satisfy inequality (2) for $f = g_f$. Hence f itself is the strongly unique best coapproximation to f. Also the strong unicity constant cannot be determined, since $\max(0,\infty)$ does not exist. Hence in the problem of strongly unique best coapproximation, it is assumed hereafter that $f \in X \setminus G$.

In contrast to best coapproximation, the strongly unique best coapproximation does not coincide with the strongly unique best approximation in inner product spaces. In the case of strongly unique best approximation, the constant $k_f > 0$ satisfying inequality (1) depends only on f. But in the case of strongly unique best coapproximation the constant $k_f > 0$ satisfying inequality (2) depends on both f and g_f.

In what follows, we suggest some counter examples for

- Strongly unique best coapproximation is not unique in general.
- Strongly unique best coapproximation is not equal to strongly unique best approximation in an inner product space.

Example 3.2. Let $X = \mathcal{R}\times\mathcal{R},\ G = \mathcal{R}\times\{0\},\ x = (0,1)$ or $x = (0,-1)$. Then the following statements are true:

(i) The point $(0,0)$ is the unique best approximation to x from G under l_1-norm, $\|(a,b)\|_1 = |a|+|b|,\ \ a,b \in \mathcal{R}$ and l_2-norm, $\|(a,b)\|_2 = \sqrt{a^2+b^2},\ \ a,b \in \mathcal{R}$.

(ii) The set $\{(b,0) : -1 \leq b \leq 1\}$ consists of best approximations to x from G under l_∞-norm, $\|(a,b)\|_\infty = \max\{|a|,\,|b|\},\ \ a,b \in \mathcal{R}$.

(iii) The point $(0,0)$ is the strongly unique best approximation to x from G under l_1-norm but it is not so under l_2-norm. In this case $k_f \leq 1$ and the strong unicity constant is 1.

(iv) The set $\{(b,0) : -1 \leq b \leq 1\}$ consists of best coapproximations to x from G under l_1-norm.

(v) The point $(0,0)$ is the unique best coapproximation to x from G under l_2-norm and l_∞-norm.

(vi) The set $\{(b,0) : -1 < b < 1\}$ consists of strongly unique best coapproximations to x from G under l_1-norm. Here $k_f = \inf\limits_{a\in\mathcal{R}} \frac{|a|+1-|a-b|}{|b|+1}$.

This shows that strongly unique best coapproximation is not unique.

(vii) The point $(0,0)$ is not a strongly unique best coapproximation to x from G under l_∞-norm.

(viii) The point $(0,0)$ is the unique strongly unique best coapproximation to x from G under l_2-norm. Here $k_f = \inf\limits_{a\in\mathcal{R}} \sqrt{a^2+1^2} - a$.

This with (iii) shows that a strongly unique best coapproximation is not equal to a strongly unique best approximation in an inner product space.

The next result answers the question:
Where is the strongly unique best coapproximation unique?

Theorem 3.3. *In an inner product space, every strongly unique best coapproximation is unique.*

Proof. Let G be a subset of an inner product space X, $f \in X \setminus G$. If g_1 and g_2 are strongly unique best coapproximations to f from G, then it is clear that g_1 and g_2 are best coapproximations to f and hence g_1 and g_2 are best approximations to f, since best coapproximation coincides with best approximation in an inner product space. Since best approximation is unique in an inner product space, $g_1 = g_2$. $\square$

Remark 3.4. Theorem 3.3 shows that the constant $k_f > 0$ satisfying inequality (2) depends only on f in an inner product space.

Let X be a normed linear space, G be a subset of X and $f \in X \setminus G$. Let $T_G(f)$ denotes the set of strongly unique best coapproximations to f from G. That is, for some $k_f > 0$,

$$T_G(f) = \{g_f \in G : \|g - g_f\| \le \|f - g\| - k_f\|f - g_f\|, \quad \text{for all} \quad g \in G\}.$$

Theorem 3.5. *Let G be a convex subset of a normed linear space X, $f \in X \setminus G$. Then the strongly unique best coapproximation to f from G is either unique or there are infinitely many. In fact, the set of strongly unique best coapproximations $T_G(f)$ forms a convex set.*

Proof. If the strongly unique best coapproximation is unique, then it is clear that $T_G(f)$ is convex. Otherwise, let g_1 and g_2 be distinct strongly unique best coapproximations to f from G. Then there exist $k_1, k_2 > 0$ such that for all $g \in G$,

$$\|g - g_1\| \le \|f - g\| - k_1\|f - g_1\|, \quad \|g - g_2\| \le \|f - g\| - k_2\|f - g_2\|.$$

To prove that $T_G(f)$ contains infinitely many elements, it is sufficient to prove that $T_G(f)$ is convex. Let $k = \min\{k_1, k_2\}$. Then for every $g \in G$, $0 \le \alpha \le 1$, it follows that

$$\begin{aligned}
&\|g-(\alpha g_1+(1-\alpha)g_2)\| \\
&\quad = \|\alpha(g-g_1)+(1-\alpha)(g-g_2)\| \\
&\quad \le \alpha\|g-g_1\|+(1-\alpha)\|g-g_2\| \\
&\quad \le \alpha\|f-g\|-\alpha k\|f-g_1\|+(1-\alpha)\|f-g\| \ -(1-\alpha)k\|f-g_2\| \\
&\quad = \|f-g\|-k(\|\alpha f-\alpha g_1\|+\|(1-\alpha)f-(1-\alpha)g_2\|) \\
&\quad \le \|f-g\|-k\|\alpha f-\alpha g_1+(1-\alpha)f-(1-\alpha)g_2\| \\
&\quad = \|f-g\|-k\|f-(\alpha g_1+(1-\alpha)g_2)\|.
\end{aligned}$$

Thus $T_G(f)$ is convex. □

In contrast to $R_G(f)$, $T_G(f)$ is not closed, when G is closed. See Example 3.2. (vi), which also shows that $T_G(f)$ is not open.

The next result establishes some more properties of $T_G(f)$.

Theorem 3.6. *Let G be a subset of a normed linear space X, $f \in X \setminus G$. Then $T_G(f)$ satisfies the following properties:*

(i) *If $g_f \in T_G(f)$, then $g_f \in T_G(\alpha^n f+(1-\alpha)^n g_f)$ for $\alpha \ge 1$ and $n = 0, 1, \cdots$.*

(ii) *$T_G(f)$ is bounded.*

If G is a subspace of X, then $T_G(f)$ satisfies the following properties:

(iii) [3] *$T_G(f+g) = T_G(f)+g$, for all $g \in G$.*

(iv) [3] *$T_G(\alpha f) = \alpha T_G(f)$, $\alpha \in \mathcal{R}$.*

Proof. (i) $g_f \in T_G(f) \Rightarrow \|g-g_f\| \le \|f-g\|-k_f\|f-g_f\|$, for all $g \in G$ and for some $k_f > 0$.

Claim. $g_f \in T_G(\alpha f+(1-\alpha)g_f)$, $\quad \alpha \ge 1$.
That is, $\|g-g_f\| \le \|\alpha f+(1-\alpha)g_f-g\|-k_f\|\alpha f-\alpha g_f\|$.
Now

$$\begin{aligned}
&\|\alpha f+(1-\alpha)g_f-g\|-k_f\|\alpha f-\alpha g_f\| \\
&\quad = \|\alpha(f-g)+(1-\alpha)(g_f-g)\|-k_f\|\alpha f-\alpha g_f\| \\
&\quad \ge \alpha\|f-g\|-(\alpha-1)\|g-g_f\|-k_f\|\alpha f-\alpha g_f\| \\
&\quad \ge \alpha(\|g-g_f\|+k_f\|f-g_f\|)-(\alpha-1)\|g-g_f\|-k_f\|\alpha f-\alpha g_f\| \\
&\quad = \|g-g_f\|.
\end{aligned}$$

Hence the claim is true. By the repeated application of the claim the result follows.

(ii) To prove that $T_G(f)$ is bounded, it is sufficient to prove for $g_f, \tilde{g_f} \in T_G(f)$ that $\|g_f-\tilde{g_f}\| < c$ for some $c > 0$, since $\|g_f-\tilde{g_f}\| < c$ implies that $\sup_{g_f,\tilde{g_f} \in T_G(f)} \|g_f -$

$\tilde{g_f}\|$ is finite. Hence the diameter of $T_G(f)$ is finite, so that $T_G(f)$ is bounded. Let $g_f \in T_G(f)$. Then there exists a constant $k_f > 0$ such that for all $g \in G$,

$$\|g - g_f\| \leq \|f - g\| - k_f\|f - g_f\|.$$

It follows that

$$\begin{aligned} \|f - g_f\| &\leq \|f - g\| + \|g - g_f\| \\ &\leq \|f - g\| + \|f - g\| - k_f\|f - g_f\| \\ &= 2\|f - g\| - k_f\|f - g_f\|. \end{aligned}$$

Thus

$$\|f - g_f\| \leq \frac{2}{1 + k_f}\|f - g\|,$$

for all $g \in G$. Hence $\|f - g_f\| \leq \frac{2}{1+k_f}d$, where $d := \inf_{g\in G}\|f - g\|$.

Claim. For $g_f, \tilde{g_f} \in T_G(f)$, $\|g_f - \tilde{g_f}\| < c$ for some $c > 0$.
Now

$$\begin{aligned} \|g_f - \tilde{g_f}\| &\leq \|g_f - f\| + \|f - \tilde{g_f}\| \\ &\leq \frac{2}{1 + k_f}d + \frac{2}{1 + k'_f}d \\ &= c, \end{aligned}$$

where k'_f is the positive constant such that

$$\|g - \tilde{g_f}\| \leq \|f - g\| - k'_f\|f - \tilde{g_f}\|,$$

for all $g \in G$, and $c = \frac{2}{1+k_f}d + \frac{2}{1+k'_f}d$. □

4. Characterization of strongly unique best coapproximation

Let X be a normed linear space, G be a subspace of X, $f \in X \setminus G$ and $g_f \in G$. Let $[G, f]$ be the subspace spanned by G and f. That is,

$$[G, f] = \{g + \alpha f : g \in G, \alpha \in \mathcal{R}\}$$

and $[G, f]^*$ be the space of all continuous linear functionals defined on $[G, f]$.
For $0 < k_f \leq 1$, define $\mathcal{L}(g_f, k_f)$ by

$$\mathcal{L}(g_f, k_f) = \{L \in [G, f]^* : L(f - g_f) = k_f\|f - g_f\| \text{ and } \|L\| = 1\}.$$

The following proposition is required to obtain a characterization of a strongly unique best coapproximation.

Proposition 4.1. *Let G be a subspace of a normed linear space X, $f \in X \setminus G$. If a best coapproximation to f from G is strongly unique, then so is the best coapproximation to every element of $[G, f]$ from G.*

Proof. Let g_f be the strongly unique best coapproximation to f from G. Then there exists a constant $k_f > 0$ such that $\|g - g_f\| \leq \|f - g\| - k_f \|f - g_f\|$.
Let $g_0 \in G$, then $g_0 + \alpha f \in [G, f]$. It is clear that $g_1 := \alpha g_f + g_0$ is a best coapproximation to $\alpha f + g_0$ from G. Hence to prove the result it is sufficient to prove that g_1 is a strongly unique best coapproximation to $\alpha f + g_0$ from G. For all $g \in G$, it follows that

$$
\begin{aligned}
\|g - g_1\| &= \|g - g_0 - \alpha g_f\| \\
&= |\alpha| \left\| \frac{g - g_0}{\alpha} - g_f \right\| \\
&\leq |\alpha| \left\{ \left\| f - \frac{g - g_0}{\alpha} \right\| - k_f \|f - g_f\| \right\} \\
&= \|\alpha f + g_0 - g\| - k_f \|\alpha f - \alpha g_f\| \\
&= \|\alpha f + g_0 - g\| - k_f \|\alpha f + g_0 - g_1\|.
\end{aligned}
$$

Thus g_1 is a strongly unique best coapproximation to $\alpha f + g_0$ from G. □

Remark 4.2. Let $f \in X \setminus G$ be an arbitrary element. Let $\tilde{f} = \frac{f - g_f}{\|f - g_f\|}$, $g_f \in G$. Hence $f = \|f - g_f\| \tilde{f} + g_f \in [G, \tilde{f}]$ and $\|\tilde{f}\| = 1$. If 0 is a strongly unique best coapproximation to $\tilde{f}$ from G, then it is clear that $\|f - g_f\| \, 0 + g_f$ is a best coapproximation to $\|f - g_f\| \tilde{f} + g_f = f$ and hence by Proposition 4.1 g_f is a strongly unique best coapproximation to f from G. Hence by Proposition 4.1, in order to prove $g_f \in G$ is a strongly unique best coapproximation to an arbitrary element $f \in X \setminus G$, it is enough to prove that 0 is a strongly unique best coapproximation to $\tilde{f} \in X \setminus G$, where $\|\tilde{f}\| = 1$. A similar fact can be proved in the context of approximation.

The following theorem is a characterization of a strongly unique best coapproximation.

Theorem 4.3. *Let G be a subspace of a normed linear space X, $f \in X \setminus G$, and $g_f \in G$. Then the following statements are equivalent:*

(i) *There exists a real number $k_f > 0$ such that for every $g \in G$,*

$$
\sup_{L \in \mathcal{L}(g_f, k_f)} L(g) = \|g\|.
$$

(ii) *The element g_f is a strongly unique best coapproximation to f from G.*

Proof. By Remark 4.2, assume without loss of generality that $g_f = 0$ and $\|f\| = 1$. Hence it is sufficient to prove that there exists a constant $k_f > 0$ such that for every

$g \in G$, $\sup_{L \in \mathcal{L}(0,k_f)} L(g) = \|g\|$ if and only if $\|g\| \le \|f-g\| - k_f$.
Assume first that there exists $k_f > 0$ such that for all $g \in G$, $\sup_{L \in \mathcal{L}(0,k_f)} L(g) = \|g\|$.
Then for every $g \in G$, it follows that

$$\begin{aligned}
\|f-g\| &= \sup_{L \in [G,f]^*;\, \|L\|=1} |L(f-g)| \\
&\ge \sup_{L \in \mathcal{L}(0,k_f)} |L(f-g)| \\
&\ge \sup_{L \in \mathcal{L}(0,k_f)} L(f-g) \\
&= \sup_{L \in \mathcal{L}(0,k_f)} (L(f) + L(-g)) = \sup_{L \in \mathcal{L}(0,k_f)} (k_f\|f\| + L(-g)) \\
&= k_f + \sup_{L \in \mathcal{L}(0,k_f)} L(-g) \\
&= k_f + \|g\|.
\end{aligned}$$

Thus $\|g\| \le \|f-g\| - k_f$ for every $g \in G$.

Conversely, assume that for every $g \in G$,

$$\|g\| \le \|f-g\| - k_f. \tag{3}$$

Let g be an arbitrary fixed element of G. Define $L' : [g,f] \to \mathcal{R}$ by

$$L'(\alpha g + \beta f) = \alpha\|g\| + \beta k_f,$$

where $\alpha, \beta \in \mathcal{R}$. It can be verified easily that L' is linear and continuous. Therefore, $L' \in [g,f]^*$.

Claim. $\|L'\| = 1$

If $\beta = 0$, then $|L'(\alpha g + \beta f)| = |L'(\alpha g)| = |\alpha\,\|g\|\,| = \|\alpha g\|$.

If $\beta \ne 0$, then

$$\begin{aligned}
|L'(\alpha g + \beta f)| &= |\alpha\,\|g\| + \beta k_f| \\
&\le |\alpha|\;\|g\| + |\beta| k_f \\
&= |\beta| \left(\frac{|\alpha|\|g\|}{|\beta|} + k_f \right) \\
&= |\beta| \left(\left\| \frac{-\alpha}{\beta} g \right\| + k_f \right) \\
&\le |\beta| \left\| f + \frac{\alpha}{\beta} g \right\| && \text{by (3)} \\
&= \|\beta f + \alpha g\|.
\end{aligned}$$

Hence $\|L'\| = \sup\{|L'(\alpha g + \beta f)| : \|\alpha g + \beta f\| \leq 1\} \leq 1$. Since $L'(\alpha g) = 1$ for $\alpha = \frac{1}{\|g\|}$, it follows that $\|L'\| = 1$.

By Hahn-Banach theorem, the continuous linear functional L' can be extended continuously and linearly to $[G, f]$ without increasing its norm. Hence assume without loss of generality that $L' \in [G, f]^*$ and that $L' \in \mathcal{L}(0, k_f)$. Since $L'(g) = \|g\|$, it follows that $\sup_{L \in \mathcal{L}(0,k_f)} L(g) \geq \|g\|$.

Since $\|L\| = 1$ for every $L \in \mathcal{L}(0, k_f)$, it follows that

$$L(g) \leq |L(g)| \leq \|L\| \, \|g\| = \|g\|.$$

Hence $\sup_{L \in \mathcal{L}(0,k_f)} L(g) \leq \|g\|$. Therefore, $\sup_{L \in \mathcal{L}(0,k_f)} L(g) = \|g\|$. □

Let G be a subspace of a normed linear space X, $f \in X \setminus G$ and $g_f \in G$. The following notation is used in the next result. Let $k_f > 0$ be such that

$$\|g - g_f\| \leq \|f - g\| - k_f \|f - g_f\|,$$

for every $g \in G$. Define a set K_{g_f} by

$$K_{g_f} = \{z \in [G, f] : L(z) \leq k_f \|f - g_f\|, \text{ for all } L \in \mathcal{L}(g_f, k_f)\}.$$

Let $L_{g_f} \in [G, f]^*$ be defined by $L_{g_f}(g + \alpha f) = \alpha k_f \|f - g_f\|$, $\alpha \in \mathcal{R}$, for every $g \in G$.

Theorem 4.4. *If g_f is a strongly unique best coapproximation to f from G, then*

$$\{z \in [G, f] : L_{g_f}(z) = k_f \|f - g_f\|\} \cap K_{g_f}$$

contains exactly one element x of the form $x = f - g_f$, where $k_f > 0$ is such that $\|g - g_f\| \leq \|f - g\| - k_f \|f - g_f\|$, for all $g \in G$.

Proof. If $x = f - g_f$, then it is clear that $x \in \{z \in [G, f] : L_{g_f}(z) = k_f \|f - g_f\|\} \cap K_{g_f}$. Assume that $x \in \{z \in [G, f] : L_{g_f}(z) = k_f \|f - g_f\|\} \cap K_{g_f}$. Then it follows that $L_{g_f}(x) = k_f \|f - g_f\|$. Hence by the definition of L_{g_f}, $x = g + f$ for some $g \in G$. Since $x \in K_{g_f}$, it follows for every $L \in \mathcal{L}(g_f, k_f)$ that

$$\begin{aligned}
& L(x) \leq k_f \|f - g_f\| \\
\Rightarrow \quad & L(g + f) \leq k_f \|f - g_f\| \\
\Rightarrow \quad & L(g + g_f - g_f + f) \leq k_f \|f - g_f\| \\
\Rightarrow \quad & L(g + g_f) + L(f - g_f) \leq k_f \|f - g_f\| \\
\Rightarrow \quad & L(g + g_f) + k_f \|f - g_f\| \leq k_f \|f - g_f\| \\
\Rightarrow \quad & L(g + g_f) \leq 0 \\
\Rightarrow \quad & \sup_{L \in \mathcal{L}(g,k_f)} L(g + g_f) \leq 0.
\end{aligned}$$

Since g_f is a strongly unique best coapproximation, by Theorem 4.3

$$\sup_{L\in\mathcal{L}(g_f,k_f)} L(g+g_f) = \|g+g_f\| \leq 0.$$

This implies that $g = -g_f$. Hence $x = f - g_f$. □

For each $g \in G$, define a set $\mathcal{L}(g, f, g_f)$ of continuous linear functionals by

$$\mathcal{L}(g, f, g_f) = \{L \in [G, f]^* : L(g - g_f) = \|g - g_f\| \text{ and } \|L\| = 1\}.$$

It is clear that $L \in \mathcal{L}(g, f, g_f)$ implies that $-L$ does not belong to $\mathcal{L}(g, f, g_f)$. Now a characterization of strongly unique best coapproximation is established.

Theorem 4.5. *Let G be a subspace of a normed linear space X, $f \in X \setminus G$ and $g_f \in G$. Then the following statements are equivalent:*

(i) *There exists a constant $k_f > 0$ such that for each $g \in G$,*

$$\sup_{L\in\mathcal{L}(g,f,g_f)} L(f) \geq k_f\|f\|.$$

(ii) *The element g_f is a strongly unique best coapproximation to f from G.*

Proof. By Remark 4.2, assume without loss of generality that $g_f = 0$ and $\|f\| = 1$. Hence it is sufficient to prove that there exists a constant $k_f > 0$ such that for every $g \in G$, $\sup_{L\in\mathcal{L}(g,f,0)} L(f) \geq k_f$ if and only if $\|g\| \leq \|f - g\| - k_f$. Assume first that there exists $k_f > 0$ such that for every $g \in G$,

$$\sup_{L\in\mathcal{L}(g,f,0)} L(f) \geq k_f.$$

Then for every $g \in G$, it follows that

$$\begin{aligned}
\|f-g\| &= \sup_{L\in[G,f]^*;\,\|L\|=1} |L(f-g)| \\
&\geq \sup_{L\in\mathcal{L}(-g,f,0)} |L(f-g)| \\
&\geq \sup_{L\in\mathcal{L}(-g,f,0)} L(f-g) \\
&= \sup_{L\in\mathcal{L}(-g,f,0)} (L(f)+L(-g)) \\
&= \sup_{L\in\mathcal{L}(-g,f,0)} (L(f)+\|g\|) \\
&= \sup_{L\in\mathcal{L}(-g,f,0)} (L(f))+\|g\| \\
&\geq k_f+\|g\|.
\end{aligned}$$

Thus $\|g\| \le \|f-g\| - k_f$, for every $g \in G$.

Conversely, assume that for every $g \in G$, $\|g\| \le \|f-g\| - k_f$.
Let g be an arbitrary but fixed element of G. Define $L' : [g,f] \to \mathcal{R}$ by $L'(\alpha g+\beta f) = \alpha\|g\| + \beta k_f$, where $\alpha, \beta \in \mathcal{R}$. By proceeding as in the proof of Theorem 4.3, it can be shown that $L' \in \mathcal{L}(g,f,0)$. Since $L'(f) = k_f$, it follows that $\sup_{L\in\mathcal{L}(g,f,0)} L(f) \ge k_f$.

□

By applying the techniques used in [1] and Theorem 4.3, a necessary and sufficient condition characterizing a strongly unique best approximation is obtained, for which the following notation is required.

For $k_f > 0$ and for each $g \in G$, define a set $\mathcal{L}(g,f,g_f,k_f)$ of continuous linear functionals by

$$\mathcal{L}(g,f,g_f,k_f) = \{L \in [G,f]^* : L(g-g_f) = k_f\|g-g_f\| \text{ and } \|L\| = 1\}.$$

Theorem 4.6. *Let G be a subspace of a normed linear space X, $f \in X \setminus G$ and $g_f \in G$. Then the following statements are equivalent:*

(i) *There exists a constant $k_f > 0$ such that for each $g \in G$,*

$$\sup_{L\in\mathcal{L}(g,f,g_f,k_f)} L(f) = \|f\|.$$

(ii) *The element g_f is a strongly unique best approximation to f from G.*

5. Some results concerning strongly unique best uniform coapproximation

Strongly unique best coapproximation (respectively, strongly unique best approximation) with respect to the uniform norm is called strongly unique best uniform coapproximation (respectively, strongly unique best uniform approximation).

Definition 5.1. The set $E(f)$ of *extreme points* of a function $f \in C[a,b]$ is defined by $E(f) = \{t \in [a,b] : |f(t)| = \|f\|_\infty\}$. For $r \in \mathcal{R}$ and $f \in C[a,b]$, define $rE(f) = \{t \in [a,b] : |f(t)| = r\|f\|_\infty\}$.

Now a condition to establish strongly unique best uniform coapproximation is obtained in the following result:

Theorem 5.2. *Let G be a subset of $C[a,b]$, $f \in C[a,b] \setminus G$ and $g_f \in G$. If there exists a constant $k_f > 0$ such that for every $g \in G$,*

$$\min_{t\in k_f E(f-g_f)} (f(t)-g_f(t))(g(t)-g_f(t)) \le -k_f\|f-g_f\|_\infty\|g-g_f\|_\infty. \tag{4}$$

Then the function g_f is a strongly unique best uniform coapproximation to f from G.

Proof. It is clear that for every $g \in G$, there exists a point $t \in k_f E(f - g_f)$ such that

$$(f(t) - g_f(t))(g(t) - g_f(t)) \leq -k_f \|f - g_f\|_\infty \|g - g_f\|_\infty.$$

This implies that for every $g \in G$,

$$|f(t) - g_f(t)|\ |g(t) - g_f(t)| \geq k_f \|f - g_f\|_\infty \|g - g_f\|_\infty.$$

Then it follows that for every $g \in G$,

$$\begin{aligned} \|f - g\|_\infty &\geq |f(t) - g(t)| \\ &= |(f(t) - g_f(t)) - (g(t) - g_f(t))| \\ &= |f(t) - g_f(t)| + |g(t) - g_f(t)| \\ &\geq |f(t) - g_f(t)| + \frac{k_f \|f - g_f\|_\infty \|g - g_f\|_\infty}{|f(t) - g_f(t)|} \\ &= k_f \|f - g_f\|_\infty + \frac{k_f \|f - g_f\|_\infty \|g - g_f\|_\infty}{k_f \|f - g_f\|_\infty}. \end{aligned}$$

Thus for every $g \in G$, $\|g - g_f\|_\infty \leq \|f - g\|_\infty - k_f \|f - g_f\|_\infty$. □

Remark 5.3. If G is a subspace of $C[a, b]$, then inequality (4) in Theorem 5.2 can be replaced by the inequality

$$\min_{t \in k_f E(f - g_f)} (f(t) - g_f(t))(g(t)) \leq -k_f \|f - g_f\|_\infty \|g\|_\infty.$$

Let A be a subset of $C[a, b]$ such that $A = \{f \in C[a, b] : f(\alpha t) = \alpha f(t), \alpha > 0\}$. Let G be a subset of $C[a, b]$ and let

$SU(G) = \{f \in C[a, b] : f$ has a strongly unique best uniform coapproximation from $G\}$.

$SU(G, 0) = \{f \in C[a, b] : f$ has 0 as its strongly unique best uniform coapproximation from $G\}$.

Then the following notion of open base for a set is required to characterize those functions which belong to $\overline{SU(G, 0)}$.

Definition 5.4. An *open base for a set* is a class of neighborhoods of the set such that each neighborhood of the set contains a neighborhood in this class.

Lemma 5.5. *Let $[a, b]$ be a closed and bounded interval in $\mathcal{R}$ such that $\alpha a \in [a, b]$ for $0 < \alpha \leq 1$. Let G be a subset of A and $f \in A \setminus G$. If there exists a constant $k_f > 0$ such that for every function $g \in G$ and every set B containing $E(f)$,*

$$\inf_{t \in B} f(k_f t) g(k_f t) \leq -k_f \|f\|_\infty \|g\|_\infty,$$

then $f \in \overline{SU(G,0)}$.

Proof. Let $\{B_n\}$ be an open base for the set $E(f)$ such that $\overline{B_n} \subset B_{n-1}$ for all n. For sufficiently large n, define

$$f_n(t) = \begin{cases} f(t), & t \in [a,b] \setminus B_{n-1} \\ \|f\|_\infty \operatorname{sgn} f(t), & t \in \overline{B_n}. \end{cases}$$

Then by Tietze extension theorem, there exists a continuous extension of f_n to $[a,b]$ such that $\|f_n\|_\infty = \|f\|_\infty$. It is clear that $\{f_n\}$ converges to f. Therefore, it is sufficient to prove that f_n has 0 as its strongly unique best uniform coapproximation from G. Let n be chosen arbitrarily large and $g \in G$. Since $\overline{B_n}$ is a set containing $E(f)$ and $\overline{B_n} \subset E(f_n)$, it follows that $\min\limits_{t \in E(f_n)} f(k_f t) g(k_f t) \leq -k_f \|f\|_\infty \|g\|_\infty$.
Since $f_n \to f$, it follows that $\min\limits_{t \in E(f_n)} f_n(k_f t) g(k_f t) \leq -k_f \|f_n\|_\infty \|g\|_\infty$.
So there exists a point $t \in E(f_n)$ such that $f_n(k_f t) g(k_f t) \leq -k_f \|f_n\|_\infty \|g\|_\infty$.
This implies that $|f_n(k_f t)|\ |g(k_f t)| \geq k_f \|f_n\|_\infty \|g\|_\infty$.

Therefore, it follows that

$$\begin{aligned} \|f_n - g\|_\infty &\geq |f_n(k_f t) - g(k_f t)| \\ &= k_f |f_n(t)| + |g(k_f t)| \\ &\geq k_f \|f_n\|_\infty + \frac{k_f \|f_n\|_\infty \|g\|_\infty}{|f_n(k_f t)|} \\ &= k_f \|f_n\|_\infty + \frac{k_f \|f_n\|_\infty \|g\|_\infty}{k_f \|f_n\|_\infty}. \end{aligned}$$

Thus for every $g \in G$, $\|g\|_\infty \leq \|f_n - g\|_\infty - k_f \|f_n\|_\infty$.
Hence f_n has 0 as its strongly unique best uniform coapproximation from G. □

Theorem 5.6. *Let* $[a,b]$ *be a closed and bounded interval in* $\mathcal{R}$ *such that* $\alpha a \in [a,b]$ *for* $0 < \alpha \leq 1$. *Let* G *be a subspace of* A, $f \in A \setminus G$ *and* $g_f \in G$. *If there exists a constant* $k_f > 0$ *such that for every function* $g \in G$ *and every set* B *containing* $E(f - g_f)$, $\inf\limits_{t \in B} (f(k_f t) - g_f(k_f t)) g(k_f t) \leq -k_f \|f - g_f\|_\infty \|g\|_\infty$,
then $f \in \overline{SU(G)}$.

Proof. By Lemma 5.5, $f - g_f \in \overline{SU(G,0)}$. Therefore, let $h_n \in SU(G,0)$ be a sequence such that h_n converges to $f - g_f$. It is clear that $h_n + g_f$ is a sequence converging to f. It is also clear that $h_n + g_f$ has g_f as its strongly unique best coapproximation from G. Hence $f \in \overline{SU(G)}$. □

Theorem 5.7. *Let* G *be a subset of a normed linear space* X. *If* G *is an existence and uniqueness set with respect to best coapproximation with a continuous cometric projection and* $SU(G) \neq \emptyset$, *then* $SU(G)$ *is an* F_σ*-set.*

Proof. For each $f \in SU(G)$, let $K(f)$ be the strong unicity constant of f. That is, $K(f)$ is the maximum of the numbers $k_f > 0$ such that for each $g \in G$,

$$\|g - g_f\| \leq \|f - g\| - k_f \|f - g_f\|,$$

where $g_f = R_G(f)$. Let $F_m = \{f \in SU(G) : K(f) \geq \frac{1}{m}\}$. It is clear that $SU(G) = \bigcup_m F_m$. Therefore, it is sufficient to prove that F_m is closed for each m. Let $\{f_n\}$ be a sequence in F_m such that $f_n \to f$. Hence it is sufficient to prove that $f \in F_m$. Let $R_G(f_n) = g_{f_n}$. Then it follows from $f_n \in F_m$ that for each $g \in G$,

$$\|g - g_{f_n}\| \leq \|f_n - g\| - \frac{1}{m}\|f_n - g_{f_n}\|. \tag{5}$$

Since the cometric projection R_G is continuous, $R_G(f_n) \to R_G(f)$. By taking limits in equation (5), it follows that $\|g - g_f\| \leq \|f - g\| - \frac{1}{m}\|f - g_f\|$, which shows that $f \in F_m$. □

6. Some relations between best uniform approximation and strongly unique best uniform coapproximation

The following result answers the question:
When does a best uniform approximation imply a strongly unique best uniform coapproximation?

Theorem 6.1. *Let G be a subset of $C[a,b]$, $f \in C[a,b] \setminus G$ and $g_f \in G$ be a best uniform approximation to f from G. If there exists a constant $k_f > 0$ such that for every function $g \in G$, $\min_{t \in E(g-g_f)} (f(t)-g(t))(g_f(t)-g(t)) \leq -k_f\|f-g_f\|_\infty\|g-g_f\|_\infty$, then the function g_f is a strongly unique best uniform coapproximation to f from G.*

Proof. For every function $g \in G$, there exists a point $t \in E(g - g_f)$ such that

$$(f(t) - g(t))(g_f(t) - g(t)) \leq -k_f\|f - g_f\|_\infty\|g - g_f\|_\infty.$$

This implies that $|f(t) - g(t)|\ |g_f(t) - g(t)| \geq k_f\|f - g_f\|_\infty\|g - g_f\|_\infty$.
Therefore, it follows that for every $g \in G$,

$$\begin{aligned}
\|f - g\|_\infty &\geq \|f - g_f\|_\infty \\
&\geq |f(t) - g_f(t)| \\
&= |(f(t) - g(t)) - (g_f(t) - g(t))| \\
&= |f(t) - g(t)| + |g_f(t) - g(t)| \\
&\geq \frac{k_f\|f - g_f\|_\infty\|g - g_f\|_\infty}{|g_f(t) - g(t)|} + |g_f(t) - g(t)| \\
&= k_f\|f - g_f\|_\infty + \|g - g_f\|_\infty.
\end{aligned}$$

Thus $\|g - g_f\|_\infty \leq \|f - g\|_\infty - k_f\|f - g_f\|_\infty$. □

Theorem 6.2. *Let G be a subset of $C[a,b]$, $f \in C[a,b] \setminus G$ and $g_f \in G$ be a best uniform approximation to f from G. If for every function $g \in G$,*

$$\min_{t \in E(g-g_f)} (f(t) - g(t))(g_f(t) - g(t)) \leq -\|f - g_f\|_\infty\|g - g_f\|_\infty,$$

then the function g_f is a strongly unique best uniform approximation and strongly unique best uniform coapproximation to f from G.

Proof. The proof is the same as that of Theorem 6.1. □

The following result shows that a best uniform coapproximation is a strongly unique best uniform approximation when a specific inequality is satisfied.

Theorem 6.3. *Let G be a subset of $C[a,b]$, $f \in C[a,b] \setminus G$ and $g_f \in G$ be a best uniform coapproximation to f from G. If there exists a constant $k_f > 0$ such that for every function $g \in G \setminus \{g_f\}$,*

$$\min_{t \in E(f-g_f)} (f(t) - g(t))(f(t) - g_f(t)) \leq -k_f \|f - g_f\|_\infty \|g - g_f\|_\infty,$$

then the function g_f is a strongly unique best uniform approximation to f from G.

Proof. There exists a point $t \in E(f - g_f)$ such that for every $g \in G \setminus \{g_f\}$,

$$(f(t) - g(t))(f(t) - g_f(t)) \leq -k_f \|f - g_f\|_\infty \|g - g_f\|_\infty.$$

This implies that for every $g \in G \setminus \{g_f\}$,

$$|f(t) - g(t)|\ |f(t) - g_f(t)| \geq k_f \|f - g_f\|_\infty \|g - g_f\|_\infty.$$

Therefore, it follows that for every $g \in G \setminus \{g_f\}$,

$$\begin{aligned} \|f - g\|_\infty &\geq \|g - g_f\|_\infty \\ &\geq |g(t) - g_f(t)| \\ &= |(g(t) - f(t)) - (g_f(t) - f(t))| \\ &= |g(t) - f(t)| + |g_f(t) - f(t)| \\ &\geq k_f \frac{\|f - g_f\|_\infty \|g - g_f\|_\infty}{|f(t) - g_f(t)|} + |g_f(t) - f(t)| \\ &= k_f \|g - g_f\|_\infty + \|f - g_f\|_\infty. \end{aligned}$$

Thus $\|f - g_f\|_\infty \leq \|f - g\|_\infty - k_f \|g - g_f\|_\infty$. □

Remark 6.4. A result similar to Theorem 6.2 can be obtained by taking $k_f = 1$ in Theorem 6.3.

The next result provides a condition to obtain a strongly unique best uniform approximation.

Theorem 6.5. *Let G be a subset of $C[a,b]$, $f \in C[a,b] \setminus G$ and $g_f \in G$. If there exists a constant $k_f > 0$ such that for every function $g \in G$,*

$$\min_{t \in k_f E(g-g_f)} (f(t) - g_f(t))(g(t) - g_f(t)) \leq -k_f \|f - g_f\|_\infty \|g - g_f\|_\infty, \tag{6}$$

then the function g_f is a strongly unique best uniform approximation to f from G.

Proof. There exists a point $t \in k_f E(g - g_f)$ such that for every $g \in G$,

$$(f(t) - g_f(t))(g(t) - g_f(t)) \leq -k_f \|f - g_f\|_\infty \|g - g_f\|_\infty.$$

This implies that for every $g \in G$,

$$|f(t) - g_f(t)|\ |g(t) - g_f(t)| \geq k_f \|f - g_f\|_\infty \|g - g_f\|_\infty.$$

Then it follows that for every $g \in G$,

$$\begin{aligned}
\|f - g\|_\infty &\geq |f(t) - g(t)| \\
&= |(f(t) - g_f(t)) - (g(t) - g_f(t))| \\
&= |f(t) - g_f(t)| + |g(t) - g_f(t)| \\
&\geq k_f \frac{\|f - g_f\|_\infty \|g - g_f\|_\infty}{|g(t) - g_f(t)|} + |g(t) - g_f(t)| \\
&= \|f - g_f\|_\infty + k_f \|g - g_f\|_\infty.
\end{aligned}$$

Thus $\|f - g_f\|_\infty \leq \|f - g\|_\infty - k_f \|g - g_f\|_\infty$. □

Remark 6.6. If G is a subspace of $C[a, b]$, then Theorem 6.5 remains true if inequality (6) is replaced by the inequality

$$\min_{t \in k_f E(g)} (f(t) - g_f(t))(g(t)) \leq -k_f \|f - g_f\|_\infty \|g\|_\infty.$$

The following Kolmogorov type criteria characterizing best uniform approximation, best uniform coapproximation, strongly unique best uniform approximation and strongly unique best uniform coapproximation are required in the sequel.

Theorem 6.7 ([14]). *Let G be a subset of $C[a, b]$ such that $\alpha g \in G$ for all $g \in G$ and $\alpha \in [0, \infty)$. Let $f \in C[a, b] \setminus G$ and $g_f \in G$. Then the following statements are equivalent:*

(i) *The function g_f is a best uniform approximation to f from G.*

(ii) *For every function $g \in G$, $\min_{t \in E(f - g_f)} (f(t) - g_f(t))\ (g(t) - g_f(t)) \leq 0$.*

Theorem 6.8 ([13]). *Let G be a subset of $C[a, b]$ such that $\alpha g \in G$ for all $g \in G$ and $\alpha \in [0, \infty)$. Let $f \in C[a, b] \setminus G$ and $g_f \in G$. Then the following statements are equivalent:*

(i) *The function g_f is a best uniform coapproximation to f from G.*

(ii) *For every function $g \in G$, $\min_{t \in E(g - g_f)} (f(t) - g_f(t))\ (g(t) - g_f(t)) \leq 0$.*

Theorem 6.9 ([16]). *Let G be a subset of $C[a,b]$ such that $\alpha g \in G$ for all $g \in G$ and $\alpha \in [0,\infty)$. Let $f \in C[a,b] \setminus G$ and $g_f \in G$. Then the following statements are equivalent:*

(i) *The function g_f is a strongly unique best uniform approximation to f from G.*

(ii) *There exists a constant $k_f > 0$ such that for every function $g \in G$,*

$$\min_{t \in E(f-g_f)} (f(t) - g_f(t))(g(t) - g_f(t)) \leq -k_f \|f - g_f\|_\infty \|g - g_f\|_\infty.$$

Theorem 6.10 ([13]). *Let G be a subset of $C[a,b]$ such that $\alpha g \in G$ for all $g \in G$ and $\alpha \in [0,\infty)$, $f \in C[a,b] \setminus G$ and $g_f \in G$. Then the following statements are equivalent:*

(i) *The function g_f is a strongly unique best uniform coapproximation to f from G.*

(ii) *There exists a constant $k_f > 0$ such that for every function $g \in G$,* $\min_{t \in E(g-g_f)} (f(t) - g_f(t))\ (g(t) - g_f(t)) \leq -k_f \|f - g_f\|_\infty\ \|g - g_f\|_\infty.$

Proposition 6.11. *Let G be a subset of $C[a,b]$, $f \in C[a,b] \setminus G$ and $g_f \in G$. Consider the following statements:*

(i) *For every function $g \in G$,*

$$\min_{t \in E(f-g_f)} (f(t) - g_f(t))(g(t) - g_f(t)) \leq -\|f - g_f\|_\infty \|g - g_f\|_\infty.$$

(ii) *For every function $g \in G$,*

$$\min_{t \in E(g-g_f)} (f(t) - g_f(t))(g(t) - g_f(t)) \leq -\|f - g_f\|_\infty \|g - g_f\|_\infty.$$

(iii) *The function g_f is a strongly unique best uniform approximation and a strongly unique best uniform coapproximation to f from G.*

Then (i) $\Rightarrow$ (iii) *and* (ii) $\Rightarrow$ (iii).

Proof. The proof follows from Theorem 6.9 and Theorem 6.10. □

Remark 6.12. Proposition 6.11 remains true if $g - g_f$ is replaced by g when G is considered as a subspace.

Recall that for $k_f > 0$, $A = \{f \in C[a,b] : f(k_f t) = k_f f(t)\}$.
Then the next result determines a condition when the same element can be a strongly unique best uniform coapproximation and a best uniform approximation.

Theorem 6.13. *Let G be a subset of A, $f \in A \setminus G$ and $g_f \in G$. If there exists a constant $k_f > 0$ such that for every function $g \in G$,*

$$\min_{t \in E(f-g_f)} (f(k_f t) - g_f(k_f t))(g(k_f t) - g_f(k_f t)) \le -k_f \|f - g_f\|_\infty \|g - g_f\|_\infty, \tag{7}$$

then the function g_f is a strongly unique best uniform coapproximation and a best uniform approximation to f from G.

Proof. By proceeding as in the proof of Theorem 5.2, it can be shown that g_f is a strongly unique best uniform coapproximation to f from G. Inequality (7) implies that $(k_f)^2 \min_{t \in E(f-g_f)} (f(t) - g_f(t))(g(t) - g_f(t)) < 0$.
Therefore, $\min_{t \in E(f-g_f)} (f(t) - g_f(t))(g(t) - g_f(t)) < 0$.
Hence by Theorem 6.7, g_f is a best uniform approximation to f from G.

By applying similar arguments used in Theorem 6.13 and using Theorem 6.5 and Theorem 6.8, the following result can be proved. □

Proposition 6.14. *Let G be a subset of A, $f \in A \setminus G$ and $g_f \in G$. If there exists a constant $k_f > 0$ such that for every function $g \in G$,*

$$\min_{t \in E(g-g_f)} (f(k_f t) - g_f(k_f t))(g(k_f t) - g_f(k_f t)) \le -k_f \|f - g_f\|_\infty \|g - g_f\|_\infty,$$

then the function g_f is a strongly unique best uniform approximation and a best uniform coapproximation to f from G.

The next result answers the question:
Under what circumstances can the same element be both a strongly unique best uniform approximation and a strongly unique best uniform coapproximation?

Proposition 6.15. *Let G be a subset of $C[a,b]$, $f \in C[a,b] \setminus G$ and $g_f \in G$. Then each of the following statements implies that the function g_f is a strongly unique best uniform approximation and strongly unique best uniform coapproximation to f from G.*

(i) *If there exists a constant $k_f > 0$ such that for every function $g \in G$,*

$$\min_{t \in E(f-g_f) \cap E(g-g_f)} (f(t) - g_f(t))(g(t) - g_f(t)) \le -k_f \|f - g_f\|_\infty \|g - g_f\|_\infty.$$

(ii) *If there exists a constant $k_f > 0$ such that for every function $g \in G$,*

$$\min_{t \in k_f E(f-g_f) \cap k_f E(g-g_f)} (f(t) - g_f(t))(g(t) - g_f(t)) \le -k_f \|f - g_f\|_\infty \|g - g_f\|_\infty.$$

Proof. This is an easy consequence of Theorems 6.9, 6.10, 5.2, and 6.5. □

Remark 6.16. If G is considered as a subspace of $C[a,b]$ or A accordingly, then Theorem 6.13 and Propositions 6.11, 6.14, 6.15 remain true when $g - g_f$ is replaced by g.

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