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On (H, μ_n) Summability of Fourier Series

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Abstract. In this paper, we have proved a theorem on Hausdorff summability of Fourier series which generalizes various known results. We prove that if

$$\int_0^t \mid \phi(u) \mid \, du \;=\; o(t) \;\text{ as }\; t \to 0 \;\text{ and }\; \lim_{n\to\infty} \int_{\pi/n}^{\eta} \frac{\mid \phi(t) - \phi(t+\pi/n) \mid}{t} \, dt \;=\; o(n)$$

where $0 < \eta < 1$, then the Fourier series is (H, μ_n) summable to s at $t = x$ where the sequence μ_n is given by

$$\mu_n \;=\; \int_0^1 x^n \chi(x) \, dx \;\; n = 0, 1, 2 \;\text{ and }\; K_n(t) \;=\; \sum_{\nu=0}^{n} \binom{n}{\nu} (\Delta^{n-\nu} \, \mu_\nu) \frac{\sin \nu t}{t}.$$

1. Definitions and notations

Let $\sum\limits_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$, the sequence-to-sequence Hausdorff transformation of the sequence $\{s_n\}$ is given by

$$t_n \;=\; \sum_{\nu=0}^{n} \binom{n}{\nu} (\Delta^{n-\nu} \, \mu_\nu) \, s_\nu \tag{1.1}$$

defines the sequence $\{t_n\}$ of Hausdorff means of the sequence $\{s_n\}$.

The $\{\mu_n\}$ form a given sequence of real or complex numbers and $(\Delta^p \, \mu_n)$ denote their difference of order p, that is, $\Delta^0 \, \mu_n \;=\; \mu_n$ and

$$\Delta^p \, \mu_n = \Delta^{p-1}\mu_n - \Delta^{p-1}\mu_{n+1} = \Delta(\Delta^{p-1} \, \mu_n), \quad (p \geq 1). \tag{1.2}$$

The series $\sum\limits_{n=0}^{\infty} a_n$ is said to be summable (H, μ_n) to the sum 's' if $\lim\limits_{n\to\infty} t_n$ exists and is equal to s, Hausdorff ([5]) discussed for the first time this method of summability systematically.

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When $\mu_n = q^n$ $(0 < q < 1)$ (1.1) reduces to

$$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} q^{\nu} (1-q)^{n-\nu} s_{\nu} \tag{1.3}$$

which is well known sequence-to-sequence Euler transformation. This method is regular for $0 < q \leq 1$. Cesàro and Hölder methods are also particular cases of Hausdorff method.

In order that the method (H, μ_n) to be conservative, it is necessary and sufficient that there should be a function $\chi(x)$, of bounded variation in [0,1] such that

$$\mu_n = \int_0^1 x^n \, d\chi(x), \quad n = 0, 1, 2. \tag{1.4}$$

We may suppose without loss of generality that $\chi(0) = 0$.

If further $\chi(x)$ satisfies the conditions that $\chi(+0) = 0$ and $\chi(1) = 1$, then μ_n is a regular moment constant and (H, μ_n) is a regular Hausdorff transformation.

The function $\chi(x)$ may have an enumerable set of discontinuities and the value of the integral (1.4) is not affected by any change in the value of $\chi(x)$ at a point of discontinuity in (0,1). In particular we may suppose that

$$\chi(x) = \frac{1}{2}\{\chi(x-0) + \chi(x+0)\} \tag{1.5}$$

for $0 < x < 1$ in which case we shall say that all discontinuities of $\chi(x)$ are normal.

If $\chi(x) = 1 - (1-x)^{\alpha}$, $\alpha > 0$, then $(H, \mu_n) \equiv (C, \alpha)$.

On the other hand, if, for $q > 0$

$$\chi(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1/(1+q) \\ 1 & \text{for } 1/(1+q) \leq x \leq 1 \end{cases} \tag{1.6}$$

then (H, μ_n) reduces to the familiar Euler-Knopp method (E, q).

Knopp and Lorentz ([6]) have proved that if (H, μ_n) defines a conservative (or regular) transformation then it also defines an absolutely conservative (or absolutely regular) transformation of the same type. Thus with a $\chi(x)$ satisfying (1.4), (H, μ_n) is an absolute convergence-preserving transformation.

2. Fourier series

Let $f(x)$ be a periodic function with period 2π, and Lebesgue integrable over $(-\pi, \pi)$ and let

$$f(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_{\nu} \cos \nu x + b_{\nu} \sin \nu x) \tag{2.1}$$

we write

$$\phi(t) \;=\; \phi_x(t) \;=\; f(x+t) + f(x-t) - 2S \tag{2.2}$$

$$\begin{cases} g_\delta^+(x) \;=\; 1/\Gamma(\delta)\; \int_0^x (x-\nu)^{\delta-1}\, g(\nu)\, d\nu, \;\; \delta > 0 \\ g_\delta^-(x) \;=\; 1/\Gamma(\delta)\; \int_x^1 (\nu-x)^{\delta-1}\, g(\nu)\, d\nu, \;\; \delta > 0 \end{cases} \tag{2.3}$$

where Γ denotes the well-known Gamma function.

These $g_\delta^+(x)$ and $g_\delta^-(x)$ are defined as the δth forward and backward fractional integrals respectively of a function $g(x)$ defined and Lebesgue integrable in (0,1). It is known that these integrals exists almost everywhere for $\delta > 0$.

And

$$K_n(t) \;=\; \sum_{\nu=0}^{n} \binom{n}{\nu} \left(\Delta^{n-\nu}\, \mu_\nu\right) \frac{\sin \nu t}{t}. \tag{2.4}$$

3. Introduction

In 1968, Sahney and Kathal ([8]) have proved the following theorem:

Theorem A. *If $\phi(t) = o(t)$ as $t \to 0+$ and*

$$\lim_{p\to\infty} \int_{\pi/p}^{\eta} \frac{\mid \phi(t) - \phi(t+\pi/p) \mid}{t} \exp\{-p(1-cost)\}\, dt \;=\; 0$$

where η is constant, then the Fourier series is summable (B) to s at the point $t = x$.

In 1989, Verma and Agarwal ([13]) have proved a theorem on almost Hausdorff summability of Fourier series. They proved the following theorems:

Theorem B. *If*

$$\int_0^t \mid \phi(u) \mid \; du \;=\; o\left(\frac{t}{(\log 1/t)^\eta}\right) \quad \text{as} \quad t \to +0$$

and

$$\int_{1/(n+p)}^{1/(n+p)^\eta} \frac{\mid \phi(u) \mid}{u}\, du \;=\; o(1) \quad \text{as} \quad n \to \infty$$

hold uniformly with respect to p, $0 < \eta < 1$, then the Fourier series is almost (H, μ_n) summable to $f(x)$ at the point $t = x$.

The object of the present paper is to extend the above theorems to (H, μ_n) summability.

4. Main theorem

We shall prove the following theorem:

Theorem. *If*

$$\int_0^t \mid \phi(t) \mid \; dt \;=\; o(t) \quad \text{as} \quad t \to 0 \tag{4.1}$$

and

$$\lim_{n\to\infty} \int_{\pi/n}^{\eta} \frac{\mid \phi(t) - \phi(t+\pi/n) \mid}{t} \, dt \;=\; o(n) \tag{4.2}$$

where $0 < \eta < 1$, *then the Fourier series* (2.1) *is* (H, μ_n) *summable to* s *at* $t = x$ *where the sequence* μ_n *is given by*

$$\mu_n \;=\; \int_0^1 x^n \; \chi(x) \; dx \quad n = 0, 1, 2 \tag{4.3}$$

and

$$K_n(t) \;=\; \sum_{\nu=0}^{n} \left(\begin{array}{c} n \\ \nu \end{array} \right) (\Delta^{n-\nu} \; \mu_\nu) \, \frac{\sin \nu t}{t}. \tag{4.4}$$

5. Lemmas

We shall require the following lemmas for the proof of the theorem:

Lemma 1. ([11], Lemma 3)

$$\sum_{\nu=1}^{n} \nu \left(\begin{array}{c} n \\ \nu \end{array} \right) x^\nu \; (1-x)^{n-\nu} \;=\; nx.$$

Lemma 2. ([7], p.589)

$$K_n(t) \;=\; o(1/t) \quad \text{for} \quad \pi/n < t \leq \eta < \pi.$$

6. Proof of the theorem

Let $h_n(x)$ be the Hausdorff transformation of the sequence $\{s_n\}$ of the partial sums of the Fourier series (2.1). Then we have ([7])

$$\begin{aligned} h_n(x) - S &= 1/\pi \textstyle\int_0^\pi \phi(t) \; K_n(t) \; dt + o(1) \\ &= 1/\pi \left(\textstyle\int_0^{\pi/n} + \int_{\pi/n}^{\eta} + \int_\eta^\pi \right) \phi(t) \; K_n(t) \; dt + o(1). \end{aligned}$$

$$h_n(x) - S \;=\; I_1 + I_2 + I_3 + o(1). \tag{6.1}$$

Now when $0 < \lambda t \leq 1$, $(\sin \lambda t)/t$ is a monotonic decreasing function of t. Hence by mean value theorem there exists a δ_n $(0 \leq \delta_n \leq 1)$ such that

$$\begin{aligned}
\mid I_1 \mid \;&\leq\; \left\{ \sum_{\nu=0}^{n} \nu \begin{pmatrix} n \\ \nu \end{pmatrix} (\Delta^{n-\nu}\, \mu_\nu)\, \int_0^{\delta_n/n} \phi(t)\, dt \right\} \\
&=\; o\left(1/n \int_0^1 \left\{ \sum_{\nu=0}^{n} \nu \begin{pmatrix} n \\ \nu \end{pmatrix} x^\nu\, (1-x)^{n-\nu} \mid d\chi(x) \mid \right\} \right) \\
&=\; o\left(\int_0^1 x \mid d\chi(x) \mid \right) \quad \text{by Lemma 1.}
\end{aligned}$$

$$\mid I_1 \mid \;=\; o(1). \tag{6.2}$$

Now

$$\begin{aligned}
I_2 \;&=\; \int_{\pi/n}^{\eta} \phi(t)\, K_n(t)\, dt \;=\; \int_{\pi/n}^{\eta} \phi(t)/t\, dt \quad \text{by Lemma 2} \\
&=\; \left(\int_{\pi/n}^{\eta-\pi/n} \left(\left\{ \frac{\phi(t) - \phi(t+\pi/n)}{t} \right\} + \frac{\pi}{n}\, \frac{\phi(t+\pi/n)}{t(t+\pi/n)} \right) dt \right. \\
&\qquad \left. - \int_0^{\pi/n} \frac{\phi(t+\pi/n)}{(t+\pi/n)}\, dt \;+\; \int_{\eta-\pi/n}^{\eta} \frac{\phi(t)}{t}\, dt \right).
\end{aligned}$$

$$I_2 \;=\; J_1 + J_2 + J_3 + J_4. \tag{6.3}$$

$$\mid J_1 \mid \;=\; o(1) \quad \text{by (4.2).} \tag{6.4}$$

$$\mid J_2 \mid \;=\; o\left(\frac{1}{n}\right) \int_{\pi/n}^{\eta-\pi/n} \frac{\phi(t+\pi/n)}{(t+\pi/n)}\, dt.$$

$$\mid J_2 \mid \;=\; o(1) \quad \text{by (4.1) and integrating by parts.} \tag{6.5}$$

$$\mid J_3 \mid \;=\; \int_0^{\pi/n} \frac{\phi(t+\pi/n)}{(t+\pi/n)}\, dt.$$

$$\mid J_3 \mid \;=\; \int_{\pi/n}^{2\pi/n} \frac{\phi(t)}{t}\, dt.$$

$$\mid J_3 \mid \;=\; o(1) \quad \text{by (4.1) and integrating by parts.} \tag{6.6}$$

Similarly

$$\mid J_4 \mid \;=\; o(1). \tag{6.7}$$

The combination of (6.4), (6.5), (6.6) and (6.7) gives us

$$I_2 = o(1). \tag{6.8}$$

Now, by regularity condition of the method, we have

$$I_3 = o(1). \tag{6.9}$$

Finally the proof of the theorem is completed by considering (6.1), (6.2), (6.8) and (6.9).

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