KYUNGPOOK Math. J. 43(2003), 489-494

On a New Theorem Involving the $\bar{H}$-function and a General Class of Polynomials

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Abstract. In this paper, we first establish an interesting theorem involving the $\bar{H}$-function introduced by Inayat - Hussain ([7], [8]). The convergence and existence condition, basic properties of this function were given by Buschman and Srivastava ([2]). Next, we obtain certain new integrals and an expansion formula by the application of our theorem. On account of the most general nature of the functions involved herein, our main findings are capable of yielding a large number of new, interesting and useful integrals, expansion formulae involving simple special functions and polynomials as their special cases. A known special case of our main theorem in also given ([11]).

1. Introduction

Recently, a finite integral involving the $\bar{H}$-function was evaluated by Gupta and Soni ([5]). The convergence and existence condition, basic properties of this function were given by Buschman and Srivastava ([2]). This function will be defined and represented in following manner ([2])

$$\begin{aligned}\text{(1.1)} \qquad \bar{H}^{M,N}_{P,Q}[z] &= \bar{H}^{M,N}_{P,Q}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N}, & (a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M}, & (b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\bar{\phi}(\xi)z^{\xi}d\xi,\end{aligned}$$

where

$$\text{(1.2)} \qquad \bar{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}$$

which contains fractional powers of some of the gamma functions. Here, and through out the paper $a_j(j = 1,\cdots,P)$ and $b_j(j = 1,\cdots,Q)$ are complex parameters, $\alpha_j \geq 0(j = 1,\cdots,P)$, $\beta_j \geq 0(j = 1,\cdots,Q)$(not all zero simultaneously) and the

Received December 4, 2001, and, in revised form, April 17, 2002.
2000 Mathematics Subject Classification: 33C45, 33C60.
Key words and phrases: $\bar{H}$-function, a general class of polynomials, expansion formulae.

exponents $A_j (j = 1, \cdots, N)$ and $B_j (j = M+1, \cdots, Q)$ can take on non-integer values.

The contour in (1.1) is imaginary axis $\mathrm{Re}(\xi) = 0$. It is suitably indented in order to avoid the singularities of the gamma functions and to keep those singularities on appropriate sides. Again, for $A_j (j = 1, \cdots, N)$ not an integer, the poles of the gamma functions of the numerator in (1.2) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j - \beta_j \xi)(j = 1, \cdots, M)$ and $\Gamma(1 - a_j + \alpha_j \xi)(j = 1, \cdots, N)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.

Evidently, when the exponents A_j and B_j all take on integral values, the $\bar{H}$-function reduces to the well known Fox's H-function ([4], [14]).

The following sufficient conditions for the absolute convergence of the defining integral for the $\bar{H}$-function have been given by Buschman and Srivastava ([2])

$$\Omega \equiv \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} A_j \alpha_j - \sum_{j=M+1}^{Q} B_j \beta_j - \sum_{j=N+1}^{P} \alpha_j > 0 \tag{1.3}$$

and

$$|\arg(z)| < \frac{1}{2}\pi\Omega, \tag{1.4}$$

where Ω is given by (1.3).

The behaviour of the $\bar{H}$-function for small values of $|z|$ follows easily from a result recently given by Rathie ([9], p.306, eq. (6.9)). We have

$$\bar{H}_{P,Q}^{M,N}[z] = O\left(|z|^{\alpha}\right), \ \ \alpha = \min_{1 \le j \le M} \left[\mathrm{Re}\left(b_j/\beta_j\right)\right], \ \ |z| \to 0. \tag{1.5}$$

Investigations of the convergence conditions, all possible types of contours, type of critical points of the integrand of (1.1), etc. can be made by an interested reader by following analogous techniques given in the well known works of Braaksma ([1]), Hai and Yakubovich ([6]). We however omit the details.

Also $S_n^m[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava ([12], p.1, eq.(1))

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \ \ n = 0, 1, 2, \cdots, \tag{1.6}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}(n,\, k \geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Lagurre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others ([13], p.158-161).

2. Main Theorem

If

$$(1-y)^{\alpha+\beta-\gamma}{}_2F_1(2\alpha,\,2\beta;\,2\gamma;\,y)=\sum_{r=0}^{\infty}a_r y^r \tag{2.1}$$

then

$$\int_0^1 {}_2F_1(\alpha,\,\beta;\,\gamma+\frac{1}{2};\,y)\;{}_2F_1(\gamma-\alpha,\,\gamma-\beta;\,\gamma+\frac{1}{2};\,y)$$
$$S_n^m[y^u]\bar{H}_{P,Q}^{M,N}\left[zy^v\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},\;(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},\;(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dy$$
$$=\sum_{k=0}^{[n/m]}\sum_{r=0}^{\infty}\frac{(-n)_{mk}}{k!}A_{n,k}\frac{(\gamma)_r}{(\gamma+\frac{1}{2})_r}a_r$$

$$\bar{H}_{P+1,Q+1}^{M,N+1}\left[z\left|\begin{matrix}(-r-uk,v;1),\;(a_j,\alpha_j;A_j)_{1,N},\;(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},\;(b_j,\beta_j;B_j)_{M+1,Q},\;(-1-r-uk,v;1)\end{matrix}\right.\right] \tag{2.2}$$

where

$$u>0,\;\;v\geq 0,\;\;-\frac{1}{2}<(\gamma-\alpha-\beta)<\frac{1}{2},\;\;v\min_{1\leq j\leq M}\left[\mathrm{Re}(b_j/\beta_j)\right]+1>0,$$

$$|arg(z)|<\frac{1}{2}\pi\Omega,\;\;\Omega\equiv\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}A_j\alpha_j-\sum_{j=M+1}^{Q}B_j\beta_j-\sum_{j=N+1}^{P}\alpha_j>0$$

and m is an arbitrary positive integer and the coefficients $A_{n,k}(n,\,k\geq 0)$ are arbitrary, real or complex.

Proof. We have ([10], p.75)

$${}_2F_1(\alpha,\,\beta;\,\gamma+\frac{1}{2};\,y)\;{}_2F_1(\gamma-\alpha,\,\gamma-\beta;\,\gamma+\frac{1}{2};\,y)=\sum_{r=0}^{\infty}\frac{(\gamma)_r}{(\gamma+\frac{1}{2})_r}a_r y^r \tag{2.3}$$

where a_r is given by (2.1).

Multiplying both sides of (2.3) by $S_n^m[y^u]\bar{H}_{P,Q}^{M,N}[zy^v]$, integrating with respect to y between the limits 0 to 1, expressing the general class of polynomials in the series form by (1.6) and the $\bar{H}$-function in terms of Mellin-Barnes contour integral by (1.1), interchanging the order of integration and evaluating the integral thus obtain with the help of a known result ([3]), we arrive at the required result after a little simplification.

3. Applications and special cases

(i) If we put $\alpha = \gamma$ in the main theorem, the value of a_r in (2.1) comes out to be equal to $(\beta)_r$ and the result (2.2) yields the following interesting integral

$$\int_0^1 {}_2F_1(\alpha,\, \beta;\, \alpha + \frac{1}{2};\, y)\ \ S_n^m[y^u]\bar{H}_{P,Q}^{M,N}[zy^v]dy$$
$$= \sum_{k=0}^{[n/m]} \sum_{r=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(\alpha)_r(\beta)_r}{(\alpha + \frac{1}{2})_r r!}$$

$$\bar{H}_{P+1,Q+1}^{M,N+1}\left[z\left|\begin{matrix}(-r-uk, v; 1),\ (a_j, \alpha_j; A_j)_{1,N},\ (a_j, \alpha_j)_{N+1,P}\\ (b_j, \beta_j)_{1,M},\ (b_j, \beta_j; B_j)_{M+1,Q},\ (-1-r-uk, v; 1)\end{matrix}\right.\right] \tag{3.1}$$

where

$$(1-y)^{-\beta} = \sum_{r=0}^{\infty}(\beta)_r y^r,\ \ u > 0,\ \ v \geq 0,\ \ \mathrm{Re}(\beta) < \frac{1}{2},$$

$$v \min_{1\leq j\leq M}\left[\mathrm{Re}(b_j/\beta_j)\right] + 1 > 0,\ \ |\arg(z)| < \frac{1}{2}\pi\Omega,\ \ \Omega > 0$$

and m is an arbitrary positive integer and the coefficients $A_{n,k}(n,\, k \geq 0)$ are arbitrary constants, real or complex.

(ii) Take $\beta = \alpha + \frac{1}{2}$ and $\alpha = -f$ (f is non-negative integer) in (3.1), we have

$$\int_0^1 (1-y)^f\ \ S_n^m[y^u]\bar{H}_{P,Q}^{M,N}[zy^v]dy \ \ = \ \ \sum_{k=0}^{[n/m]} \sum_{r=0}^{f} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(-f)_r}{r!}$$

$$\bar{H}_{P+1,Q+1}^{M,N+1}\left[z\left|\begin{matrix}(-r-uk, v; 1),\ (a_j, \alpha_j; A_j)_{1,N},\ (a_j, \alpha_j)_{N+1,P}\\ (b_j, \beta_j)_{1,M},\ (b_j, \beta_j; B_j)_{M+1,Q},\ (-1-r-uk, v; 1)\end{matrix}\right.\right] \tag{3.2}$$

where

$$u > 0,\ \ v \geq 0,\ \ v \min_{1\leq j\leq M}\left[\mathrm{Re}(b_j/\beta_j)\right] + 1 > 0,\ \ |\arg(z)| < \frac{1}{2}\pi\Omega,\ \ \Omega > 0$$

and m is an arbitrary positive integer and the coefficients $A_{n,k}(n,\, k \geq 0)$ are arbitrary constants, real or complex.

(iii) Now evaluating the integral on the left of (3.2) with the help of a know result ([3]), we establish the following interesting expansion formula

$$\sum_{k=0}^{[n/m]} \sum_{r=0}^{f} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(-f)_r}{r!}$$
$$\bar{H}^{M,N+1}_{P+1,Q+1}\left[z\left|\begin{matrix}(-r-uk,v;1),\ (a_j,\alpha_j;A_j)_{1,N},\ (a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},\ (b_j,\beta_j;B_j)_{M+1,Q},\ (-1-r-uk,v;1)\end{matrix}\right.\right]$$
$$= \Gamma(f+uk+1) \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}$$

$$\bar{H}^{M,N+1}_{P+1,Q+1}\left[z\left|\begin{matrix}(0,v;1),\ (a_j,\alpha_j;A_j)_{1,N},\ (a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},\ (b_j,\beta_j;B_j)_{M+1,Q},\ (-1-f-uk,v;1)\end{matrix}\right.\right] \tag{3.3}$$

provided that both sides exist.

If we take $n=0$(the polynomials $S_0^m[y]$ will reduce to $A_{0,0}$ and can be taken to be unity without loss of generality), $\alpha_j = \beta_j = A_j = B_j = 1$ for all j, the main theorem (2.2) reduce to a known theorem obtained by Srivastava ([11], p.236, eq.(1.2)). Also, on taking $n=0$ and $\alpha_j = \beta_j = A_j = B_j = 1$ for all j, the results (3.1), (3.2) and (3.3) reduce to the known results obtained by Srivastava ([11], p. 237, eqs. (1.5), (1.6) and (1.7)).

Acknowledgement. The author is thankful to Prof. H. M. Srivastava, Principal M. Raisinghani and Prof. K. C. Gupta for their kind help and encouragement. The author is also thankful to the worthy referee for making useful suggestions for the improvement of the paper.

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