

# Routley–Meyer Semantics for E-R, Ee-R, Eec-R

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## Abstract

In this paper we investigate the relevance logic E-R of the Entailment E without the reductio (R), and its extensions Ee-R, Eec-R: Ee-R is the E-R with the expansion (e) and Eec-R the Ee-R with the chain (c). We give completeness for each E-R, Ee-R, and Eec-R by using Routley–Meyer semantics.

## 1. Introduction

Among relevance systems, E of Entailment has been considered important from early on in the study of relevance logics. Note that the favorite of Anderson and Belnap: they are the beginners of relevance logics (see [1]). Most of all, this system is important in the sense that the strict implication, i.e., so called entailment, shows ‘necessity’ and ‘relevance’ which are the necessary and sufficient conditions of logical implication. Note that the strict implication of modal logic just shows the ‘necessity’ and that the relevant implication of R just the ‘relevance’. (cf. [1])

Many (algebraic) papers say that this system has the same connectives as the classical logic with respect to conjunction, disjunction, and negation (see [5]). However, exactly speaking (from the algebraic point of view), this is not true. When we consider ‘De Morgan lattices’ (or ‘quasi-Boolean algebras’) that characterize E with respect to those connectives, we find that E has one unnatural axiom scheme, i.e., the reductio (R), in the

sense that this is not the part characterized by those algebras.<sup>1)</sup>

Thus, in [14] we investigated some neighbors of the relevance **T** of Ticket Entailment without the reductio (R), i.e., the **T** w (T-R), and its extensions the T-R with the expansion (e) (Te-R), the Te-R with the chain (c) (Tec-R). We also gave completeness for these systems by using Routley-Meyer (RM) semantics. Note that in it we already explained why relevance systems without R, e.g., T-R, can be formally (or algebraically) interesting; that **E** is not decidable and that the decidability of **EW**, i.e., the **E** without the contraction (W), still remains as an open question (see [3, 4, 8, 13]). Note also that e is important (and useful) in decidability and c is important when we consider any system to be many-valued logic, especially infinite-valued logic, (consider the intuitionist propositional logic **H** and the Dummett's LC of its infinite-valued extension that can be regarded as the **H** with c).

We, however, postpone to investigate the decidability of the neighbors of **E** with e and c (as many-valued logics) to another occasion because we need too much space to complete it in this paper. Instead, we just show that the same idea as in [14] can be applied to the system **E**. Namely, we investigate the **E** without R (E-R), the E-R with e (Ee-R), and the Ee-R with c (Eec-R). Note that Routley and Meyer [10, 11] investigated RM semantics for Positive Entailment **E**<sup>+</sup> and **E** and gave completeness for each system. However, they did not do that for

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1) To understand the meaning of "characterize" (or "characteristic"), see this passage in [7]: "Given a matrix  $M$ , let  $\text{Taut}(M)$  (the "tautologies of  $M$ ") be the set of sentences that take designated values for every interpretation in underlying algebra. Given a unary assertional logic  $L$ , a matrix  $M$  is said to be characteristic for  $L$  whenever for all sentences  $\phi$ ,  $\vdash_L \phi$  iff  $\phi \in \text{Taut}(M)$ ."

the above E-systems, and we have not yet found any literature that investigated them. So, we give RM semantics for each E-R, Ee-R, and Eec-R, and thus its completeness. This means that the neighbors of **E** with e and c can be semantically complete.

For convenience, by  $E(ec)\text{-}R$  we shall ambiguously express E-R, Ee-R, Eec-R all together, if we do not need distinguish them, but context should determine which system is intended; often by  $E(e)\text{-}R$ , just E-R and Ee-R. Depending on the works of Routley and Meyer, and Dunn in [2, 5, 10, 11, 12], we can show the completeness for  $E(ec)\text{-}R$ . We shall also adopt the similar notation, terminology, and results found in them, and assume familiarity with them.

## 2. Axiom Schemes and Rules for $E(ec)\text{-}R$

For convenience, we present just the axiom schemes and the inference for  $E(ec)\text{-}R$ . For the remainder we shall follow the cu notation and terminology. The formation of  $E(ec)\text{-}R$  can be given following list of axiom schemes and rules:

### AXIOM SCHEMES

- A1.  $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$  (suffixing)
- A2.  $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$  (contraction)
- A3.  $((A \rightarrow A) \rightarrow B) \rightarrow B$  (specialized assertion)
- A4.  $(A \wedge B) \rightarrow A, (A \wedge B) \rightarrow B$  ( $\wedge$ -elimination)
- A5.  $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$  ( $\wedge$ -introduction)
- A6.  $A \rightarrow (A \vee B), B \rightarrow (A \vee B)$  ( $\vee$ -introduction)
- A7.  $((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$  ( $\vee$ -elimination)
- A8.  $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$  (distributive law)

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- A9.  $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$       where  $\Box A := (A \rightarrow A) \rightarrow A$   
A10.  $\sim\sim A \rightarrow A$       (classical double negation)  
A11.  $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$       (contraposition)  
A12.  $(A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow B))$       (expansion)  
A13.  $(A \rightarrow B) \vee (B \rightarrow A)$       (chain)

### RULES

- $A \rightarrow B, A \vdash B$       (modus ponens (MP))  
 $A, B \vdash A \wedge B$       (adjunction (AD))

### DEFINITION

- df1.  $A \circ B := \sim(A \rightarrow \sim B)$

### SYSTEMS

- E-R.      A1 - A11;  
Ee-R.      E-R + A12;  
Eec-R.      Ee-R + A13.

## 3. Routley-Meyer frames and models for E(ec)-R

Following [2, 5, 7], calling relevant model structures *Routley-Meyer (RM) frames*, we define an *(RM) frame*. A frame is a structure  $\mathbf{S} = (U, \sqsubseteq, R, Z, \cdot)$ , where  $(U, \sqsubseteq, R, Z)$  is a left assertional frame<sup>2)</sup> and  $\cdot$  is a unary operation on  $U$ , such that

2) That is,  $U$  is a set,  $Z (\subseteq U)$  is a left lower identity ( $Z \circ A \subseteq A$ ) satisfying the following Ili

- (Ili)  $\exists \zeta, \in Z, (R\zeta a\beta)$  iff  $a \subseteq \beta$ ,  
 $R \subseteq U^3$ , and  $\sqsubseteq$  is a partial-order satisfying:  
 $Ra\beta\gamma$  &  $\alpha' \subseteq a$  imply  $R\alpha'\beta\gamma$ ,  
 $Ra\beta\gamma$  &  $\beta' \subseteq \beta$  imply  $Ra\beta'\gamma$ ,  
 $Ra\beta\gamma$  &  $\gamma' \subseteq \gamma$  imply  $Ra\beta\gamma'$ .

the following definitions and postulates hold:<sup>3)</sup> ( $\zeta \in Z$ )

- df2.  $a \sqsubseteq \beta := \exists \zeta (R\zeta a\beta)$   
 df3.  $R^2 a\beta\gamma\delta := \exists x (Ra\beta x \ \& \ R\gamma\delta)$   
 df4.  $R^2 a(\beta\gamma)\delta := \exists x (Rax\delta \ \& \ R\beta\gamma x)$   
 df5.  $Sa := \forall x, y (Raxy \Rightarrow \exists \zeta (R\zeta xy))$

(With respect to the following postulates, just for convenience, to represent some  $\zeta$  we take  $\theta$ , which Routley and Meyer take in their semantics. Note that  $\theta$ , by which we represent some  $\zeta$  ( $\in Z$ ), itself is a member of  $Z$ , i.e.,  $\theta \in Z$ .)<sup>4)</sup>

- p0.  $Ra\beta\gamma$  and  $a' \sqsubseteq a$  imply  $Ra'\beta\gamma$  (monotonicity)  
 p1.  $R^2 a\beta\gamma\delta \Rightarrow R^2 \beta(a\gamma)\delta$   
 p2.  $Ra\beta\gamma \Rightarrow R^2 a\beta\beta\gamma$   
 p3.  $\forall a \exists x (Sx \ \text{and} \ Raxa)$   
 p4.  $Raaa$  (idempotence)  
 p5.  $Ra\beta\gamma \Rightarrow Rav'\beta'$   
 p6.  $a'' = a$   
 p7.  $Ra\beta\gamma \Rightarrow \exists x (Rx\beta a \ \& \ Ra\beta\gamma)$   
 p8.  $R\theta a\beta$  or  $R\theta \beta a$

For E-R, df1 - df5 plus p0 - p6:

For Ee-R, definitions and postulates of E-R + p7:

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More exactly to understand a left assertional frame, see [7]. Note that  $U$  is expressed as  $K$  in [5] (as well as in [11, 12]); and that, for convenience, we take a left lower identity in place of a right lower one, which Dunn and Hardegee take in [7].

3) Note that we take df5 for the modal character of E (see [2]).

4) Often, in proofs of section 4 and 5, by  $\theta$  we shall also ambiguously represent some  $\zeta$ , if we do not need distinguish them, but context should determine what is intended.

For Eec-R, definitions and postulates of Ee-R + p8.

Note that each postulate for E-R, i.e., p1 to p6, is those in [2, 5, 11]; that p7 and p8 is those in [14]; that we have p0 following [5, 7]; also that  $\subseteq$  is a partial order (p.o.) on U with respect to E(e)-R, i.e., E-R and Ee-R, and a linear order (l.o.) on U with respect to Eec-R. Following Dunn (and Hardegree) [6] (and [7]), we regard U as a set of "states of information", and for  $\alpha, \beta \in U$ ,  $\alpha \subseteq \beta$  means that the information of  $\alpha$  is included in that of  $\beta$ .

By a *model* for E(ec)-R, we mean a structure  $\mathbf{M} = (U, \subseteq, R, Z, \cdot, \models)$ , where  $(U, \subseteq, R, Z, \cdot)$  is a frame and  $\models$  is a relation from U to sentences of E(ec)-R satisfying the following conditions:

(Atomic Hereditary Condition (AHC))

for a propositional variable p, if  $\alpha \models p$  and  $\alpha \subseteq \beta$ , then  $\beta \models p$ ;  
 (Evaluation Clauses (EC)) for formulas A, B

( $\wedge$ )  $\alpha \models A \wedge B$  iff  $\alpha \models A$  and  $\alpha \models B$ ;

( $\vee$ )  $\alpha \models A \vee B$  iff  $\alpha \models A$  or  $\alpha \models B$ ;

( $\rightarrow$ )  $\alpha \models A \rightarrow B$  iff for all  $\beta, \gamma$ , if  $R\alpha\beta\gamma$  and  $\beta \models A$ , then  $\gamma \models B$ ;

( $\sim$ )  $\alpha \models \sim A$  iff  $\alpha \not\models A$ .

A formula A is *true* on  $v$  at  $\alpha$  of U just in case  $\alpha \models A$ ; A is *verified* on  $\mathbf{M}$  in case  $\zeta$  (especially  $\emptyset$ ),  $\in Z$ ,  $\models A$ ; A *entails* B on  $\mathbf{M}$  in case  $\forall x \in U$ , if  $x \models A$ , then  $x \models B$ ; A *E(ec)-R-entails* B just in case A entails B in every model; and A is *E(ec)-R-valid* in a frame  $\mathbf{S}$  just in case it is verified in all evaluations therein.

Let  $\Sigma$  be the class of frames. A sentence  $A$  is E(ec)-R-valid, in symbols  $\models_{\text{E(ec)-R}} A$ , if and only if  $\forall S \in \Sigma$ ,  $A$  is E(ec)-R-valid in  $S$ .

#### 4. Soundness for E(ec)-R

Following [2, 5], we give the soundness for E(ec)-R. To prove it, we need the Verification Lemma below. First, by an induction on  $A$ , we can easily prove

**Lemma 1** (Hereditary Condition (HC)) For any formula  $\models A$  and  $\alpha \sqsubseteq \beta$ , then  $\beta \models A$ ).

Since with respect to the connectives  $\sim, \wedge, \vee, \rightarrow$ , we have the same evaluations as in [2, 5, 12], we can use the Verification Lemma in them. Thus,

**Lemma 2** (Verification Lemma)  $A$  entails  $B$  on  $v$  only if  $A \rightarrow B$  is verified, i.e., true at  $\zeta$  ( $\in Z$ ), on  $v$ . Thus,  $A$  entails  $B$  in a given model  $\mathbf{M} = (U, \sqsubseteq, R, Z, \cdot, \models)$ , only if  $A \rightarrow B$  is E(ec)-R-valid in the model: that is, for every  $x$  ( $\in U$ ) if  $x \models A$  then  $x \models B$  only if  $\zeta \models A \rightarrow B$ . And  $A$  E(ec)-R-entails  $B$  only if  $A \rightarrow B$  is E(ec)-R-valid.

**Proof:** By Lemma 2 and 3 in [12] and definitions. (Using Lemma can also prove this, see the Verification Lemma in [2, 5]).  $\square$

Let  $\vdash_{\text{E(ec)-R}} A$  be the theoremhood of  $A$  in E(ec)-R. We note that each postulate was used in [2, 5, 11, 12, 14]. Thus, the Soundness for E(ec)-R is immediate.

**Proposition 1 (Soundness)**      If  $\vdash_{E(ec)-R} A$ , then  $\models_{E(ec)-R} A$ .

**Proof:** We just prove that each instance of the axiom scheme valid in all frames, i.e.,  $E(ec)$ - $R$ -valid, as an example. For A12, it suffices by Lemma 2 to assume that  $\alpha \models A \rightarrow B$  and show  $\alpha \models A \rightarrow (A \rightarrow B)$ . To show this last, we assume that p7 holds. First, by the assumption and  $(\rightarrow)$ , we can assume that  $R\alpha\beta$  and  $\beta \models A$  only if  $\beta \models B$ . Then, by p7, there is  $x$  such that  $Rx\beta\alpha$  and  $R\alpha\beta$ , and thus using  $(\rightarrow)$ , we obtain  $x \models A \rightarrow (A \rightarrow B)$  from the assumptions. From this, we get that  $\beta \models A$  only if  $\alpha \models A \rightarrow B$  by  $Rx\beta\alpha$ , and thus by  $R\alpha\beta$   $\beta \models A$  only if  $\beta \models B$ , as desired.

With respect to each instance of the other axiom schemes, we note this: from Lemma 4 in [12], it follows that each instance of the conjunction, disjunction, and negation axiom schemes, i.e., A4 to A8, A10, and A11, is  $E(ec)$ - $R$ -valid: by Lemma 5 in [11] and those of § 48.6 in [2], each instance of A1 to A3 and A9 is  $E(ec)$ - $R$ -valid: by Proposition 1 in [14], each instance of A13 is  $Eec$ - $R$ -valid.

From Proposition 1 in [14], it follows that each rule, i.e., preserves  $E(ec)$ - $R$  validity.  $\square$

## 5. Completeness for $E(ec)$ - $R$

We give the completeness for  $E(ec)$ - $R$  by using the well-known Henkin-style proofs for modal logic, but with prime theories in place of maximal theories. To do this, we define some theories. We interpret  $\vdash_{E(ec)-R}$  as the deducibility consequence relation of the logic  $E(ec)$ - $R$ . By an  $E(ec)$ - $R$ -theory, we mean a set  $T$  of sentences closed under deducibility, i.e., closed under MP and

AD: by a *prime E(ec)-R-theory*, a theory  $\mathbf{T}$  such that if  $A \vee B \in \mathbf{T}$ , then  $A \in \mathbf{T}$  or  $B \in \mathbf{T}$ ; and by a *trivial E(ec)-R theory*, the entire set of sentences of E(ec)-R. As Dunn states in Remark 4 in [6], we note that an E(ec)-R-theory  $\mathbf{T}$  contains all of the theorems of E(ec)-R. Thus it is what has been called a "regular theory" in the relevance logic literature. That is, by an E(ec)-R-theory we mean a regular E(ec)-R-theory. This means that  $\mathbf{T}$  is never empty. In the results below, there is no role either for trivial E(ec)-R theories. Hence, by an "E(ec)-R theory" we mean a non-trivial one.

Let a *canonical E(ec)-R-frame* be a structure  $\mathbf{S} = (U_{\text{can}}, \sqsubseteq_{\text{can}}, R_{\text{can}}, Z_{\text{can}}, \dot{\text{can}})$ , where  $\sqsubseteq_{\text{can}}$  is an information order on  $U_{\text{can}}$ ,  $Z_{\text{can}}$  is a set of any prime E(ec)-R theory, i.e.,  $\zeta_{\text{can}} (\in Z_{\text{can}})$ ,  $Z_{\text{can}} \subseteq U_{\text{can}}$ ,  $U_{\text{can}}$  is the set of prime E(ec)-R theories extending  $\zeta_{\text{can}}$ ,  $R_{\text{can}}$  is  $R$  below restricted to  $U_{\text{can}}$ .

(1)  $R\alpha\beta\vee$  iff for any formula  $A, B$  of E(ec)-R, if  $A \rightarrow B \in \alpha$  and  $A \in \beta$ , then  $B \in \vee$ ,

and  $\dot{\text{can}}$  is  $*$  restricted to  $U_{\text{can}}$ . We call a frame *fitting* for E(ec)-R if for each axiom scheme of E(ec)-R the corresponding semantical postulate holds. Where  $\alpha$  is a prime theory, let  $\dot{\alpha}$  be the set of every formula  $A$  such that  $\sim A$  does not belong to  $\alpha$ , i.e.,  $\dot{\alpha} = \{A: \sim A \notin \alpha\}$ .

As we mentioned above, we take the ideas of proofs from the Henkin-style completeness proofs. Thus, note that the base  $\mathbf{0}_{\text{can}}$ , i.e.,  $\mathbf{0}$ , among  $\zeta_{\text{can}} (\in Z_{\text{can}})$ , is constructed as a prime E(ec)-R-theory that excludes nontheorems of E(ec)-R, i.e., excludes  $A$  such that not  $\vdash_{\text{E(ec)-R}} A$ . Note also that in proofs

below, by  $0$ , i.e.,  $0_{\text{can}}$ , we often represent  $\zeta_{\text{can}}$  (as well as  $0$ ) if context can make clear what is intended (cf. see section 3). The partial orderedness and the linear orderedness of a canonical E(ec)-R-frame depend on  $*$  restricted on  $U_{\text{can}}$ . Then, first, it is obvious that

**Proposition 2** A canonical E(ec)-R-frame is partially ordered.

**Proposition 3** For E(ec)-R, a canonical frame is connected (and thus linearly ordered).

**Proof:** By Proposition 3 in [14].  $\square$

**Proposition 4** The canonically defined E(ec)-R-frame is a fitting for E(ec)-R.

**Proof:** As an example we just prove that p7 holds. (Note that to prove the other postulates it is enough for us to point out Theorem 1 of section 48.3, and 48.6 in [2], Lemma 6 in [11], Lemma 13 in [12], and Proposition 4 in [14].)

For p7, we assume that  $R\alpha\beta\gamma$ . We need to show that there is a prime theory  $x$  such that  $Rx\beta\alpha$  and  $R\alpha\beta\gamma$ . Suppose  $A \rightarrow B \in \alpha$  and  $A \in \beta$ . Then,  $A \rightarrow (A \rightarrow B) \in \alpha$  by A12. Let us take  $\alpha$  to be  $x$ . Then, by the assumptions, we obtain  $R\alpha\beta\alpha$  and  $R\alpha\beta\gamma$ . That is, by the assumptions  $A \rightarrow (A \rightarrow B) \in \alpha$ ,  $A \in \beta$ , we get  $A \rightarrow B \in \alpha$ , and by the assumptions  $A \rightarrow B \in \alpha$ ,  $A \in \beta$ , we obtain  $B \in \gamma$ , as desired. Since  $\alpha$  is a prime theory, it ensures that there is a prime theory that satisfies p7.  $\square$

Next, we need to define an appropriate relation  $\models$  on  $\mathbf{S} = (U_{\text{can}}, \subseteq_{\text{can}}, R_{\text{can}}, Z_{\text{can}}, \cdot_{\text{can}})$ . We define it to be that

$$a \models A \text{ iff } A \in a.$$

However, we need to verify that this satisfies AHC and EC. Note that since the positive part of E(ec)-R satisfies Definiteness section 42.1 in [2], we can directly use Fact 1 and Fact 2 of section 48.3 in [2], which are considered for  $\mathbf{R}^+$ , and thus we can use Theorem 2 of the same section.

**Proposition 5** The canonically defined  $(U_{\text{can}}, \subseteq_{\text{can}}, R_{\text{can}}, Z_{\text{can}}, \models_{\text{can}})$  is indeed an E(ec)-R model.

**Proof:** By Proposition 5 in [14].  $\square$

Thus,  $(U_{\text{can}}, \subseteq_{\text{can}}, R_{\text{can}}, Z_{\text{can}}, \models_{\text{can}})$  is an E(ec)-R model. So, since, by construction,  $\mathbf{0}$  excludes our chosen nontheorem  $A$  and the canonical definition of  $\models$  agrees with membership, we can state that for each nontheorem  $A$  of E(ec)-R, there is an E(ec)-R model  $a$  in which  $A$  is not  $\mathbf{0} \models A$ . It gives us the (weak) completeness for E(ec)-R as follows.

**Theorem 1** (Weak Completeness) If  $\models_{\text{E(ec)-R}} A$ , then  $\vdash_{\text{E(ec)-R}} A$ .

Next, let us prove the strong completeness for E(ec)-R. As  $\mathbf{R}^+$  in [2], we define  $A$  to be an *E(ec)-R consequence* of a set of formulas  $\Gamma$  if and only if for every E(ec)-R model, whenever  $a \models B$  for every  $B \in \Gamma$ ,  $a \models A$ , for (not just  $\mathbf{0}$  but) all  $a \in U$ . Let us say that  $A$  is *E(ec)-R deducible* from  $\Gamma$  if and only if  $A$  is in every E(ec)-R theory containing  $\Gamma$ . Then,

**Proposition 6** If  $\Gamma \not\models_{\text{E(ec)-R}} A$ , then there is a prime theory that  $\Gamma \subseteq \zeta$  and  $A \notin \zeta$ .

**Proof:** Take an enumeration  $\{A_n: n \in \omega\}$  of the well-formed formulas of  $E(ec)\text{-}R$ . We define a sequence of sets by induction as follows:

$$\zeta_0 = \{A': \Gamma \not\vdash_{E(ec)\text{-}R} A'\}.$$

$$\zeta_{i+1} = \text{Th}(\zeta_i \cup \{A_{i+1}\}) \text{ if it is not the case that } \zeta_i, A_{i+1} \vdash_{E(ec)\text{-}R} A, \\ \zeta_i \quad \text{otherwise.}$$

Let  $\zeta$  be the union of all these  $\zeta_n$ 's. It is easy to see that theory not containing  $A$ . Also we can show that it is a prime.

Suppose toward contradiction that  $B \vee C \in \zeta$  and  $B, C \notin \zeta$ . Then the theories obtained from  $\zeta \cup B$  and  $\zeta \cup C$  must both contain  $A$ . It follows that there is a conjunction of members of  $\zeta$   $\zeta'$  such that  $\zeta' \wedge B \vdash_{E(ec)\text{-}R} A$  and  $\zeta' \wedge C \vdash_{E(ec)\text{-}R} A$ . Note that if  $\vdash_{E(ec)\text{-}R} A \rightarrow B$ , then  $A \vdash_{E(ec)\text{-}R} B$ . Then, by A8 and MP, we get  $(\zeta' \wedge B) \vee (\zeta' \wedge C) \vdash_{E(ec)\text{-}R} A$ . And we obtain  $\zeta' \wedge (B \vee C) \vdash_{E(ec)\text{-}R} A$  by the prefixing (as a theorem), A8, and MP. From this it follows that  $A \in \zeta$ , which is contrary to our supposition.  $\square$

Thus, by using Propositions 5, 6 we can show its strong completeness as follows.

**Theorem 2 (Strong Completeness)** If  $\Gamma \models_{E(ec)\text{-}R} A$ , then  $\Gamma \vdash_{E(ec)\text{-}R} A$ .

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