*Journal of Korean*
*Data & Information Science Society*
*2003, Vol. 14, No.4 pp. 1091~1102*

# A Note on Estimating Parameters in The Two-Parameter Weibull Distribution

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Abstract

The Weibull variate is commonly used as a lifetime distribution in reliability applications. Estimation of parameters is revisited in the two-parameterWeibull distribution. The method of product spacings, the method of quantile estimates and the method of least squares are applied to this distribution. A comparative study between a simple minded estimate, the maximum likelihood estimate, the product spacings estimate, the quantile estimate, the least squares estimate, and the adjusted least squares estimate is presented.

KeyWords : *Least squares estimate; Maximum likelihood estimate; Method of product spacings estimate; Quantile estimate; Simple minded estimate.*

Running Title : Weibull Parameter Estimation

## 1. Introduction

The random variable X has a Weibull distribution with two parameters $\beta$ and $\eta$ if it has a probability density function of the form:

$$f(x) = \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-\left(\frac{x}{\eta}\right)^{\beta}} \ ; \quad \beta > 0,\ \eta > 0 \tag{1}$$

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The distribution function of the Weibull distribution (1) can be written as

$$F(x) = 1 - e^{-\left(\frac{x}{\eta}\right)^{\beta}} \; ; \quad \beta > 0, \; \eta > 0 \tag{2}$$

The random variables $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ are defined as an ordered random sample from the Weibull distribution (1).

In the literature, estimation of parameters in the Weibull distribution is discussed extensively. Readers are referred to the following references: Engelhardt(1975), Harter (1965), Harter and Moore (1965), and Stone (1977). In this paper, three new methods of parameter estimation are introduced. The method of product spacings, the method of quantile estimates and the method of adjusted least squares. These methods are compared to various existing methods.

In the following section (Section 2) different estimation procedures are presented, such as, a method labeled 'the simple minded estimate' (SME), the maximum likelihood estimation method (MLE), the method of maximum product spacings (MPS), the method of quantile estimation (QE), the least squares method(LSE), and the adjusted least squares method (ALS) are discussed. In Section 3, a comparison study is conducted using simulation. In Section 4, a concluding summary is presented. In Section 5, a brief acknowledgement is added.

## 2. Estimation Procedures

### 2.1. Simple Minded Estimates (SME)

By investigating (2), it can be easily seen that $P(W \leq \eta) = 1 - e^{-1} = 0.6321$, for every $\beta$. Hence, the $63.21^{th}$ percentile can be used as an estimate for $\eta$, say $\hat{\eta}_S$. And, since $\beta$ is the power parameter, smaller values are preferred. Let us consider the minimum of the sample $X_{1:n}$ as $\hat{\beta}_S$ , the simple minded estimate for $\beta$.

### 2.2. Obtaining the Maximum Likelihood Estimates (MLE)

If $X_1, X_2, \cdots, X_n$ are independent random variables each having the probability density function (1), then the maximum likelihood estimators of $\beta$ and $\eta$ are the solutions of the following likelihood equations:

$$\frac{\sum_{i=1}^{n} x_i^{\hat{\beta}_L} \ln X_i}{\sum_{i=1}^{n} X_i^{\hat{\beta}_L}} - \frac{1}{\hat{\beta}_L} - \frac{\sum_{i=1}^{n} \ln X_i}{n} = 0 \tag{3}$$

and

$$\hat{\eta}_L = \left( \frac{\sum_{i=1}^{n} X_i^{\hat{\beta}_L}}{n} \right)^{1/\hat{\beta}_L} . \tag{4}$$

The solutions can easily be obtained using the Newtom-Raphson method with $\hat{\beta}s$ and $\hat{\eta}s$ as the startion values.

### 2.3. Applying the Method of Product Spacings (MPS)

The method of product spacings (MPS) was concurrently introduced by Cheng and Amin (1983) and Ranneby (1984). Let

$$D_i = \int_{x_{i-1}}^{x_i} f(x;\theta)\,dx, \quad i = 1, 2, \cdots, n+1,$$

where $x_0$ is the lower limit and $x_{n+1}$ is the upper limit of the domain of the density function $f(x;\theta)$, and $\theta$ can be vector-valued. Clearly, the spacings sum to unity, $\sum D_i = 1$. The MPS method is, quite simply, to choose $\theta$ to maximize the geometric mean of the spacings,

$$G = \left( \prod_{i=1}^{n+1} D_i \right)^{\frac{1}{n+1}}$$

or, equivalently, its logarithm

$$H = \ln G.$$

MPS estimation gives consistent estimators under much more general conditions than MLEs. MPS estimators are asymptotically normal and are asymptotically as efficient as MLEs when these exist. For detailed goodness properties of MPS estimators, readers are referred to Cheng and Amin (1983), Ranneby (1984), Cheng and Iles (1987), Shah and Gokhale (1993), and the references therein.

Using the density function (1) and the cdf (2), H can be written as follows:

$$H = \frac{1}{n+1} \left[ \ln\left\{ 1 - e^{-\left(\frac{x_{1:n}}{\eta}\right)^{\beta}} \right\} + \sum_{i=1}^{n-1} \ln\left\{ e^{-\left(\frac{x_{i:n}}{\eta}\right)^{\beta}} - e^{-\left(\frac{x_{i+1:n}}{\eta}\right)^{\beta}} \right\} - \left( \frac{X_{n:n}}{\eta} \right)^{\beta} \right]. \tag{5}$$

By maximizing (5) for different values of $\beta$ and $\eta$, the MPS estimates can be obtained as $\hat{\beta}_P$ and $\hat{\eta}_P$. The Newton-Raphson method can be used in solving the two differential equations. The equations are not displayed here because they are tedious. The simple minded estimates (SME) are used as the starting values.

### 2.4. Finding the Quantile Estimates (QE)

Methods of estimation which are based on using the quantiles of the corresponding distributions are denoted as Quantile Estimates (QE). Quandt (1966) found that the performance of quantile estimates were not markedly inferior to maximum likelihood estimates. On occassions they might be preferable because of their resistance to outliers. Thomas (1976) cast doubts on such observations. Recently, Schmid (1997) considered a variation of percentile estimators known as Elemental Estimators for the three-parameter weibull distribution and Castillo and Hadi (1995) considered quantiles of continuous random variables in estimating their parameters. Readers are referred to these two references and the references therein for historical background and for other details. The quantile estimate (QE) in general can be summarized as follows.

Let $\theta = \{\theta_1, \theta_2, \cdots, \theta_r\}$ be the parameters to be estimated and $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics obtained from a random sample from $F(x;\theta)$, where, for fixed $\theta$, $F(x;\theta)$ is assumed to be strictly increasing on the interior of its support. Also, let $I = \{i_1, i_2, \cdots, i_r\}$ be a set of $r$ distinct indices, where $i_j \in \{1, 2, \cdots, n\}$, and $j = \{1, 2, \cdots, r\}$. Then, one can write

$$F(x_{i:n};\theta) \cong p_{i:n}, \quad i \in I$$

that is,

$$x_{i:n} \cong F^{-1}(p_{i:n};\theta), \ i \in I, \tag{6}$$

where, $p_{i:n} = (i-1)/(n+b)$ is an empirical distribution of $F(x_{i:n};\theta)$ or suitable plotting positions, and $a$ and $b$ are constants. The values of $a$ and $b$ are chosen(either theoretically or based on simulation) so that the resulting estimators have certain desirable properties (e.g., minimum root mean square error). Replacing the approximation by equality in (6), will result in a set of $r$ independent equations in $r$ unknowns, $\theta_1$, $\theta_2$, $\cdots$, $\theta_r$. An elemental estimate of $\theta$ can then be obtained by solving (6) for $\theta$. Note that these elemental estimates are based on the percentile method.

The estimates obtained from (6) depend on $r$ observations. A subset of $r$ observations is known as an elemental subset and the resultant estimate is known as an elemental estimate of $\theta$. Thus, from a sample of size $n$, there are $nCr$ elemental estimates. For large $n$ and $r$, the number of elemental subsets may be too large for the computation of all elemental estimates to be feasible. In such cases, instead of computing all possible elemental estimates, one may select a pre-specified number, $N$, of elemental subsets either systematically, based on some theoretical considerations, or at random. For each of these subsets, an elemental estimate of $\theta$ is computed and this collection of estimates is denoted by $\hat{\theta}_{j1}$, $\hat{\theta}_{j2}$, $\cdots$, $\hat{\theta}_{jN}$ , $j = 1, 2, \cdots r$. These elemental estimates can then be combined, using some suitable (preferably robust) function, to obtain an overall final estimate of $\theta$. A commonly used robust function is the median (MED), as shown below,

$$\hat{\theta} = median(\hat{\theta}_{j1}, \hat{\theta}_{j2}, \cdots, \hat{\theta}_{jN}).$$

The estimates are unique even when the method of moments (MOM) and the MLE equations have multiple solutions or when the MOM and the MLE do not exist.

### 2.4.1. The Two-parameter Weibull Distribution

In the two-parameterWeibull distribution (1), using the cdf (2), the $p$th quantile is given by

$$q(p;\beta;\eta) = e^{\ln\eta + \frac{1}{\beta}(\ln(-\ln(1-p)))}; \; 0 < p < 1.$$

There are two parameters, so two equations are needed. Assuming $I = \{i, j\}$ the equation (6) can be represented as

$$x_{i:n} = e^{\ln\eta + \frac{1}{\beta}(\ln(-\ln(1-p_{i:n})))},$$

$$x_{j:n} = e^{\ln\eta + \frac{1}{\beta}(\ln(-\ln(1-p_{j:n})))},$$

where $i < j$. It follows that the quantile estimates of $\beta$ and $\eta$ are given by

$$\hat{\beta}_{ij} = \frac{\ln\left(\dfrac{\ln(1-p_{i:n})}{\ln(1-p_{j:n})}\right)}{\ln(x_{i:n}) - \ln(x_{j:n})}$$

and

$$\hat{\eta}_{ij} = \frac{X_{i:n}}{(-\ln(1-p_{i:n}))^{\frac{1}{\beta_{ij}}}}.$$

After choosing values for the $p_{i:n}$, $i = 1, 2, \cdots, n$, overall estimates for $\beta$ and $\eta$ are obtained as

$$\hat{\beta}_Q = median(\hat{\beta}_{ij})$$

and

$$\hat{\eta}_Q = median(\hat{\eta}_{ij}),$$

where Q stands for quantile estimate. In absence of having the true quantiles, the empirical quantiles,

$$p_{i:n} = \frac{i}{n+1}$$

are suggested.

### 2.5. Using the Least Squares Method (LSM)

Algebraically, it can be easily seen from (2) that

$$\ln\ln\frac{1}{1-F(X_{i:n})} = \beta\ln X_{i:n} - \beta\ln\eta. \tag{7}$$

Using (7) and $F(X_{i:n}) = \frac{i}{n+1}$, Al-Fawzan (2000) suggested least squares estimates of $\beta$ and $\eta$ as

$$\hat{\beta}_L = \frac{\sum_{i=1}^{n}(V_i - \overline{V})W_i}{\sum_{i=1}^{n}(V_i - \overline{V})^2} \tag{8}$$

where $V_i = \ln X_{i:n}$, $W_i = \ln\ln\frac{1}{1-\frac{i}{n+1}}$ and $\overline{V} = \frac{1}{n}\sum_{i=1}^{n}\ln X_{i:n}$ and

$$\hat{\eta}_L = e^{\hat{V} - \overline{W}/\hat{\beta}_L}, \tag{9}$$

where $\overline{W} = \frac{1}{n}\sum_{i=1}^{n}\ln\ln\frac{1}{1-\frac{i}{n+1}}$.

### 2.6. The Adjusted Least Squares Estimates (ALS)

In (7), if we substitute $F(X_{i:n}) = \frac{i}{n+1}$, then for a given sample size the left side is fixed, hence the equation (7) can be re-written as

$$\ln X_{i:n} = \frac{1}{\beta}\ln\ln\frac{1}{1-F(X_{i:n})} + \ln\eta = \frac{1}{\beta}\ln\ln\frac{1}{1-\frac{1}{n+1}} + \ln\eta. \tag{10}$$

From this, the adjusted least square estimates of $\beta$ and $\eta$ are given by

$$\hat{\beta}_A = \frac{\sum_{i=1}^{n}(W_i - \overline{W})^2}{\sum_{i=1}^{n}(W_i - \overline{W})V_i} \tag{11}$$

and

$$\hat{\eta}_A = e^{\overline{V} - \overline{W}/\hat{\beta}_A} \tag{12}$$

where $V_i$, $W_i$, $\overline{V}$ and $\overline{W}$ are defined in Section 6.

## 3. Simulation Results

In this section the SME, MLE, MPS, QE, LSE, and ALS estimates (as described in Sections 2 through 7) are compared using Monte Carlo simulation. The results depend on the sample size $n$ and on the values of the parameters. Using $\beta = 0.5$, 1.0, 2.0 and $\eta = 0.5$, 1.0, 2.0, 1000 ($m$) random samples of sizes $n$ = 20, 50, 100 from (1) are generated. For each estimate, the bias is computed as

$$BIAS(\hat{\beta}) = \frac{1}{m}\sum_{k=1}^{m}\hat{\beta}_k - \beta$$

and

$$BIAS(\hat{\eta}) = \frac{1}{m}\sum_{k=1}^{m}\hat{\eta}_k - \eta.$$

The root mean-squared error (RMSE) is calculated using

$$RMSE(\hat{\beta}) = \sqrt{\frac{1}{m}\sum_{k=1}^{m}(\hat{\beta}_k - \beta)^2}$$

and

$$RMSE(\hat{\eta}) = \sqrt{\frac{1}{m}\sum_{k=1}^{m}(\widehat{\eta_k - \eta})^2}.$$

The average absolute difference between the true and estimated distribution functions is defined as

$$D_{abs} = \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}|\, F(x_{ij};\beta,\eta) - F(x_{ij};\hat{\beta},\hat{\eta})\,| \,.$$

The average of the maximum absolute difference between the true and estimated distribution function within each sample is defined as

$$D_{\max} = \frac{1}{m}\sum_{j=1}^{m} \max_{i} \mid F(x_{ij};\beta,\eta) - F(x_{ij};\hat{\beta},\hat{\eta}) \mid .$$

The measures $D_{abs}$ and $D_{\max}$ are overall measures which are useful, especially in cases of vector-valued parameters. The results of the simulation are included in Table 1.

Table 1: Simulation Results

| | BIAS($\hat{\eta}$) | RMSE($\hat{\eta}$) | BIAS($\hat{\beta}$) | RMSE($\hat{\beta}$) | $D_{abs}$ | $D_{\max}$ |
|---|---|---|---|---|---|---|
| $\eta=0.5$ $\beta=0.5$ | | | | | | |
| $n=20$ | | | | | | |
| SME | −0.024024 | 0.162010 | 0.093632 | 0.362101 | 0.074604 | 0.123659 |
| MLE | 0.037713 | 0.114543 | 0.037520 | 0.259470 | 0.058445 | 0.095007 |
| MPS | −0.030877 | 0.093229 | 0.065505 | 0.275623 | 0.059303 | 0.092178 |
| QE | 0.000822 | 0.105409 | 0.063802 | 0.302417 | 0.061760 | 0.098793 |
| ALS | −0.027173 | 0.107205 | 0.065183 | 0.278748 | 0.061042 | 0.096721 |
| LSE | −0.050887 | 0.113362 | 0.106048 | 0.314732 | 0.063557 | 0.100535 |
| $n=50$ | | | | | | |
| SME | −0.031815 | 0.131082 | 0.041084 | 0.206989 | 0.054322 | 0.091703 |
| MLE | 0.013067 | 0.059600 | 0.016180 | 0.152725 | 0.034888 | 0.056296 |
| MPS | −0.021780 | 0.058514 | 0.027937 | 0.157295 | 0.035702 | 0.056934 |
| QE | −0.001809 | 0.063419 | 0.028048 | 0.176455 | 0.038140 | 0.061793 |
| ALS | −0.021435 | 0.071198 | 0.031587 | 0.162208 | 0.037753 | 0.061715 |
| LSE | −0.035907 | 0.077353 | 0.053775 | 0.177812 | 0.039429 | 0.064828 |
| $n=100$ | | | | | | |
| SME | −0.032997 | 0.113815 | 0.021172 | 0.140692 | 0.043129 | 0.073575 |
| MLE | 0.007276 | 0.041261 | 0.008366 | 0.105960 | 0.025070 | 0.040569 |
| MPS | −0.013325 | 0.041273 | 0.014491 | 0.107686 | 0.025433 | 0.040836 |
| QE | −0.001424 | 0.044210 | 0.015060 | 0.119348 | 0.027053 | 0.044071 |
| ALS | −0.015193 | 0.052176 | 0.019003 | 0.111465 | 0.027174 | 0.044787 |
| LSE | −0.024453 | 0.056690 | 0.032246 | 0.120105 | 0.028337 | 0.046932 |
| $\eta=1.0$ $\beta=1.0$ | | | | | | |
| $n=20$ | | | | | | |
| SME | −0.046659 | 0.324890 | 0.043934 | 0.301742 | 0.074684 | 0.123883 |
| MLE | 0.075426 | 0.229087 | 0.008626 | 0.240389 | 0.058445 | 0.095008 |
| MPS | −0.061754 | 0.186458 | 0.035088 | 0.246645 | 0.059303 | 0.092178 |
| QE | 0.001644 | 0.210818 | 0.028247 | 0.266655 | 0.061760 | 0.098793 |
| ALS | −0.054345 | 0.214410 | 0.034197 | 0.248942 | 0.061042 | 0.096721 |
| LSE | −0.101774 | 0.226725 | 0.070040 | 0.268354 | 0.063557 | 0.100535 |
| $n=50$ | | | | | | |
| SME | −0.063474 | 0.262230 | 0.022290 | 0.190649 | 0.054339 | 0.091749 |
| MLE | 0.026135 | 0.119200 | 0.005405 | 0.146810 | 0.034888 | 0.056296 |
| MPS | −0.043561 | 0.117028 | 0.016860 | 0.148854 | 0.035702 | 0.056934 |
| QE | −0.003619 | 0.126837 | 0.014269 | 0.166007 | 0.038140 | 0.061793 |
| ALS | −0.042870 | 0.142397 | 0.019944 | 0.152602 | 0.037753 | 0.061715 |
| LSE | −0.071814 | 0.154707 | 0.040568 | 0.162523 | 0.039429 | 0.064828 |

Table 1: Simulation Results(Continued)

| | BIAS($\hat{\eta}$) | RMSE($\hat{\eta}$) | BIAS($\hat{\beta}$) | RMSE($\hat{\beta}$) | $D_{abs}$ | $D_{\max}$ |
|---|---|---|---|---|---|---|
| $n=100$ | | | | | | |
| SME | -0.065976 | 0.227631 | 0.012053 | 0.134252 | 0.043128 | 0.073573 |
| MLE | 0.014551 | 0.082522 | 0.002887 | 0.104679 | 0.025070 | 0.040569 |
| MPS | -0.026650 | 0.082546 | 0.008930 | 0.105459 | 0.025433 | 0.040836 |
| QE | -0.002849 | 0.088421 | 0.008322 | 0.116093 | 0.027053 | 0.044071 |
| ALS | -0.030386 | 0.104353 | 0.013138 | 0.108307 | 0.027174 | 0.044787 |
| LSE | -0.048906 | 0.113381 | 0.025714 | 0.114293 | 0.028337 | 0.046932 |
| $\eta=2.0 \quad \beta=2.0$ | | | | | | |
| $n=20$ | | | | | | |
| SME | -0.091980 | 0.650662 | 0.021069 | 0.290175 | 0.074737 | 0.124002 |
| MLE | 0.150853 | 0.458173 | -0.005677 | 0.239185 | 0.058445 | 0.095008 |
| MPS | -0.123508 | 0.372917 | 0.020625 | 0.240527 | 0.059303 | 0.092178 |
| QE | 0.003288 | 0.421636 | 0.011247 | 0.260748 | 0.061760 | 0.098793 |
| ALS | -0.108691 | 0.428820 | 0.019444 | 0.242913 | 0.061042 | 0.096721 |
| LSE | -0.203547 | 0.453449 | 0.053743 | 0.255333 | 0.063557 | 0.100535 |
| $n=50$ | | | | | | |
| SME | -0.126796 | 0.524525 | 0.013268 | 0.186857 | 0.054348 | 0.091771 |
| MLE | 0.052269 | 0.238399 | 0.000074 | 0.145848 | 0.034888 | 0.056296 |
| MPS | -0.087122 | 0.234056 | 0.011481 | 0.146657 | 0.035702 | 0.056934 |
| QE | -0.007238 | 0.253674 | 0.007580 | 0.163566 | 0.038140 | 0.061793 |
| ALS | -0.085741 | 0.284792 | 0.014325 | 0.149928 | 0.037753 | 0.061715 |
| LSE | -0.143627 | 0.309414 | 0.034383 | 0.157295 | 0.039429 | 0.064828 |
| $n=100$ | | | | | | |
| SME | -0.131932 | 0.455264 | 0.007604 | 0.132616 | 0.043128 | 0.073573 |
| MLE | 0.029102 | 0.165043 | 0.000143 | 0.104758 | 0.025070 | 0.040569 |
| MPS | -0.053300 | 0.165091 | 0.006171 | 0.105067 | 0.025433 | 0.040836 |
| QE | -0.005698 | 0.176841 | 0.004992 | 0.115414 | 0.027053 | 0.044071 |
| ALS | -0.060772 | 0.208706 | 0.010249 | 0.107493 | 0.027174 | 0.044787 |
| LSE | -0.097812 | 0.226761 | 0.022563 | 0.112277 | 0.028337 | 0.046932 |

## 4. Summary and Concluding Remarks

From Table 1, it is observed that all the estimates seem to be consistent as the RMSE decreases when the sample size increases. The estimates appear to be asymptotically unbiased as the BIAS decreases when the sample size increases except in case of SME in estimating $\eta$. Both $D_{abs}$ and $D_{\max}$ decrease uniformly when the sample size increases for all cases. For a clear comparison, the rankings (smallest to largest) are given in Table 2. Here we see the performances as in the order (superior to inf erior) of MLE, MPS, QE, ALS, LSE, and SME. It should also be noted that: for smaller samples MLE in estimating $\eta$ are poor; bias is the smallest in estimating $\eta$ in case of QE except for $n$ = 100, $\beta$ = 0.5, $\eta$ = 0.5; SME is worst by far in comparison with other estimates; ALS is better throughout in comparison with LSE. This study is beneficial in light of its inclusion of newer estimation procedures. In conclusion, MLE should be used if large sample properties are desirable and ALS should be used if the sample size is small and for computational convenience.

Table 2: Simulation Rankings

| | BIAS($\hat{\eta}$) | RMSE($\hat{\eta}$) | BIAS($\hat{\beta}$) | RMSE($\hat{\beta}$) | $D_{abs}$ | $D_{\max}$ |
|---|---|---|---|---|---|---|
| $\eta = 0.5$ $\beta = 0.5$ | | | | | | |
| $n = 20$ | | | | | | |
| SME | 2 | 6 | 5 | 6 | 6 | 6 |
| MLE | 5 | 5 | 1 | 1 | 1 | 2 |
| MPS | 4 | 1 | 4 | 2 | 2 | 1 |
| QE | 1 | 2 | 2 | 4 | 4 | 4 |
| ALS | 3 | 3 | 3 | 3 | 3 | 3 |
| LSE | 6 | 4 | 6 | 5 | 5 | 5 |
| $n = 50$ | | | | | | |
| SME | 5 | 6 | 5 | 6 | 6 | 6 |
| MLE | 2 | 2 | 1 | 1 | 1 | 1 |
| MPS | 4 | 1 | 2 | 2 | 2 | 2 |
| QE | 1 | 3 | 3 | 4 | 4 | 4 |
| ALS | 3 | 4 | 4 | 3 | 3 | 3 |
| LSE | 6 | 5 | 6 | 5 | 5 | 5 |
| $n = 100$ | | | | | | |
| SME | 6 | 6 | 5 | 6 | 6 | 6 |
| MLE | 1 | 1 | 1 | 1 | 1 | 1 |
| MPS | 3 | 2 | 2 | 2 | 2 | 2 |
| QE | 2 | 3 | 3 | 4 | 3 | 3 |
| ALS | 4 | 4 | 4 | 3 | 4 | 4 |
| LSE | 5 | 5 | 6 | 5 | 5 | 5 |
| $\eta = 1.0$ $\beta = 1.0$ | | | | | | |
| $n = 20$ | | | | | | |
| SME | 2 | 6 | 5 | 6 | 6 | 6 |
| MLE | 5 | 5 | 1 | 1 | 1 | 2 |
| MPS | 4 | 1 | 4 | 2 | 2 | 1 |
| QE | 1 | 2 | 2 | 4 | 4 | 4 |
| ALS | 3 | 3 | 3 | 3 | 3 | 3 |
| LSE | 6 | 4 | 6 | 5 | 5 | 5 |
| $n = 50$ | | | | | | |
| SME | 5 | 6 | 5 | 6 | 6 | 6 |
| MLE | 2 | 2 | 1 | 1 | 1 | 1 |
| MPS | 4 | 1 | 3 | 2 | 2 | 2 |
| QE | 1 | 3 | 2 | 5 | 4 | 3 |
| ALS | 3 | 4 | 4 | 3 | 3 | 4 |
| LSE | 6 | 5 | 6 | 4 | 5 | 5 |

Table 2: Simulation Rankings(Continued)

| | BIAS($\hat{\eta}$) | RMSE($\hat{\eta}$) | BIAS($\hat{\beta}$) | RMSE($\hat{\beta}$) | $D_{abs}$ | $D_{\max}$ |
|---|---|---|---|---|---|---|
| | | | $n = 100$ | | | |
| SME | 6 | 6 | 4 | 6 | 6 | 6 |
| MLE | 2 | 1 | 1 | 1 | 1 | 1 |
| MPS | 3 | 2 | 3 | 2 | 2 | 2 |
| QE | 1 | 3 | 2 | 5 | 3 | 3 |
| ALS | 4 | 4 | 5 | 3 | 4 | 4 |
| LSE | 5 | 5 | 6 | 4 | 5 | 5 |
| | | | $\eta = 2.0 \quad \beta = 2.0$ | | | |
| | | | $n = 20$ | | | |
| SME | 2 | 6 | 5 | 6 | 6 | 6 |
| MLE | 5 | 5 | 1 | 1 | 1 | 2 |
| MPS | 4 | 1 | 4 | 2 | 2 | 1 |
| QE | 1 | 2 | 2 | 5 | 4 | 4 |
| ALS | 3 | 3 | 3 | 3 | 3 | 3 |
| LSE | 6 | 4 | 6 | 4 | 5 | 5 |
| | | | $n = 50$ | | | |
| SME | 5 | 6 | 4 | 6 | 6 | 6 |
| MLE | 2 | 2 | 1 | 1 | 1 | 1 |
| MPS | 4 | 1 | 3 | 2 | 2 | 2 |
| QE | 1 | 3 | 2 | 5 | 4 | 4 |
| ALS | 3 | 4 | 5 | 3 | 3 | 3 |
| LSE | 6 | 5 | 6 | 4 | 5 | 5 |
| | | | $n = 100$ | | | |
| SME | 6 | 6 | 4 | 6 | 6 | 6 |
| MLE | 2 | 1 | 1 | 1 | 1 | 1 |
| MPS | 3 | 2 | 3 | 2 | 2 | 2 |
| QE | 1 | 3 | 2 | 5 | 3 | 3 |
| ALS | 4 | 4 | 5 | 3 | 4 | 4 |
| LSE | 5 | 5 | 6 | 4 | 5 | 5 |

## 5. Acknowledgement

The authors would like to thank the editor for facilitating the publication of the paper. The authors also would like to thank the referees for their useful comments which improved the presentation of the paper significantly.

## References

1. AL-FAWZAN, M. A. Methods for estimating the parameters of the Weibull distribution. Interstat, October 2000, 1-11.
2. CASTILLO, E. and HADI, A. S. (1995). A method for estimating

parameters and quantiles of distributions of continuous random variables. Computational Statistics & Data Analysis, 20, 421-439.

3. CHENG, R. C. H. and AMIN, N. A. K. (1983). Estimating Parameters in Continuous Univariate Distributions with a Shifted Origin. Journal of Royal Statistical Society, Series B, 45(3), 394-403.
4. CHENG, R. C. H. and ILES, T. C. (1987). Corrected Maximum Likelihood in Non-Regular Problems. Journal of Royal Statistical Society, Series B, 49(1), 95-101.
5. ENGELHARDT, M. (1975). On simple estimation of the parameters of the Weibull or extreme-value distribution. Technometrics, 17(3), 369-374.
6. HARTER, H. L. (1965). Point and interval estimators based on order statistics for the scale parameter of a Weibull population with known shape parameter. Technometrics, 7(3), 405-422.
7. HARTER, H. L. and MOORE, A. H. (1965). Maximum likelihood estimation of the parameters of Gamma and Weibull populations from complete and from censored samples. Technometrics, 7(4), 639-643.
8. QUANDT, R. E. (1966). Old and new methods of estimation and the Pareto distribution. Metrika, 10, 55-82.
9. RANNEBY, B. (1984). The Maximum Spacing Method. An Estimation Method Related to the Maximum Likelihood Method. Scandanavian Journal of Statistics, 11, 93-112.
10. SCHMID, U. (1997). Percentile estimators for the three-parameter weibull distribution for use when all parameters are unknown. Communications in Statistics - Theory and Methods, 26(3), 765-785.
11. SHAH, A. and GOKHALE, D. V. (1993). On Maximum Product of Spacings (MPS) Estimation for Burr XII Distributions. Communications in Statistics - Simulation and Computation. 22(3), 615-641.
12. STONE, G. C. and HEESWIJK, G. V. (1977). Parameter estimation for the Weibull distribution. IEEE Trans. on Elect. Insul., EI-12(4), 253-260. THOMAS, D. L. (1976). Reciprocal moments of linear combinations of exponential variates. Journal of the American Statistical Association, 71, 506-512.

[ received date : Aug. 2003, accepted date : Oct. 2003 ]