Journal of Korean
Data & Information Science Society
2003, Vol. 14, No.4 pp. 1045~1053

Markovian EWMA Control Chart for Several Correlated Quality Characteristics

Duk-Joon Chang[1) · Yong-Man Kwon[2) · Yeon-Woong Hong[3)

Abstract

Markovian EWMA control chart for simultaneously monitoring mean vector of the several correlated quality characteristics is investigated. Properties of multivariate Shewhart chart and EWMA chart are evaluated for matched FSI (fixed sampling interval) and VSI(variable sampling interval) scheme. We obtained VSI EWMA chart is more efficient than Shewhart chart for small or moderate shifts. And, we obtained stablized numerical results with Markov chain method when the number of transient state is greater than 100.

Keywords : Markovian control chart, control statistic, average time to signal

1. Introduction

The Markov chain approach for the RL(run length) distribution of a discrete one-sided CUSUM chart was originally developed by Brook and Evans(1972). Woodall(1983) presented a method of Markov chain approach for approximating the RL distribution of one-sided CUSUM procedures for continuous random variables. Cui and Reynolds(1988) considered VSI Shewhart $\overline{X}$-chart with runs rules using Markov chain. Lucas and Saccucci(1990) evaluated the properties of an EWMA chart to monitor the mean of a normally distributed process.

Markov chain method and integral equation method have traditionally been used to obtain aymptotic numerical properties of the univariate CUSUM and EWMA

1) Professor, Dept. of Statistics, Changwon National University, Changwon, 641-773, Korea.
E-mail : djchang@sarim.changwon.ac.kr

2) Associate Professor, Dept. of Computer Science and Statistics, Chosun University, Kwangju, 501-759, Korea.

3) Associate Professor, Division of Intrenet Industrial Engineering, Dongyang University, Yongju, Kyungpook, 750-012, Korea.

control charts which have continuous chart statistic. Compared to the Markov chain method, the integral equation method usually provides great accuracy for the same computational effort. But, the Markov chain method is relatively easy to use and offers more flexibility for computing some quantities that can not be easily obtained with the integral equation method. In recent years, the Markov chain approach has become increasingly popular.

To monitor mean vector of correlated quality characteristics, we evaluated the properties of multivariate EWMA control chart for the matched FSI and VSI scheme with Markov chain approach in this paper.

2. Evaluating Control Statistic

Assume that the process of interest has $p\,(p \geq 2)$ quality characteristics represented by the random vector $\underline{X} = (X_1, \cdots, X_p)'$ and we obtain a sequence of random vectors $\underline{X}_1, \underline{X}_2, \cdots,$ where $\underline{X}_i = (\underline{X}'_{i1}, \cdots, \underline{X}'_{in})'$ is a sample of observations at the sampling time $i\,(i = 1, 2, \cdots)$ and $\underline{X}_{ij} = (\underline{X}_{ij1}, \cdots, \underline{X}_{ijp})'$. It will be assumed that the successive observation vectors between and within samples are distributed independent multivariate normal distribution with $N_p(\underline{\mu}, \Sigma)$. Let $\underline{\theta}_0 = (\underline{\mu}_0, \Sigma_0)$ be the known target values for $\underline{\theta} = (\underline{\mu}, \Sigma)$.

Because a control chart can be viewed as repeated tests of significance, we can obtain multivariate control statistic for monitoring $\underline{\mu}$ by using the likelihood ratio test(LRT) statistic for testing $H_0: \underline{\mu} = \underline{\mu}_0$ vs $H_1: \underline{\mu} \neq \underline{\mu}_0$ where Σ_0 is known.

Likelihood ratio λ at the ith sample can be expressed as

$$\lambda = \exp\left[-\frac{n}{2} (\overline{\underline{X}}_i - \underline{\mu})' \Sigma_0^{-1} (\overline{\underline{X}}_i - \underline{\mu}) \right]. \tag{2.1}$$

Let Z_i^2 be $-2\ln\lambda$. Then

$$Z_i^2 = n(\overline{\underline{X}}_i - \underline{\mu})' \Sigma_0^{-1} (\overline{\underline{X}}_i - \underline{\mu}). \tag{2.2}$$

Thus, the statistic Z_i^2 under $H_0: \underline{\mu} = \underline{\mu}_0$ can be used as the control statistic for monitoring $\underline{\mu}$ of p related quality variables. Alt(1982) described various types of multivariate Shewhart type T^2 charts based on Hotelling's T^2 statistic and provided recommendations for implemenation.

Since the statistic Z_i^2 has a chi-square distribution with p degrees of freedom, the percentage point of Z_i^2 can be obtaind from a chi-square distribution. When the process has shifted to μ from the target μ_0, Z_i^2 has a non-central chi-square distribution with p degrees of freedom and noncentrality parameter $\tau^2 = n(\mu - \mu_0)' \Sigma_0^{-1} (\mu - \mu_0)$.

3. Shewhart Chart

Shewhart chart, one of most widely used control chart, has a good ability to detect large changes in monitored parameter quickly and is easy to implement the process. However, the Shewhart chart is slow to signal small or moderate changes in the process parameters.

The null hypothesis $H_0 : \mu = \mu_0$ will be rejected if $Z_i^2 > \chi^2_{1-\alpha}(p)$. Thus, the LRT statistic Z_i^2 can be used as Shewhart control statistic for μ. Hence a multivariate Shewhart chart based on Z_i^2 is

$$Z_i^2 = n(\overline{X}_i - \mu_0)' \Sigma_0^{-1} (\overline{X}_i - \mu_0). \tag{3.1}$$

This chart signals whenever $Z_i^2 > \chi^2_{1-\alpha}(p)$. Reynolds(1995) showed that the optimal VSI uses only two sampling intervals d_1, d_2 spaced as apart as possible. For the two sampling intervals VSI chart bases on Z_i^2, suppose the sampling interval ;

$$d_1 \text{ is used when } Z_i^2 \in (g, h]$$
$$d_2 \text{ is used when } Z_i^2 \in (0, g].$$

The parameters g, h can be obtained from chi-square distribution to guarantee a desired average time to signal(ATS) and average number of samples to signal(ANSS).

4. Markovian EWMA Chart

Traditional FSI control charts take samples from a process at fixed time interval. Recently, the application of VSI control charts has become quite frequent. To design and evaluate performances of the multivariate FSI and VSI control charts, it is very convenient when the distribution of control statistic is known. When the control statistic is continuous, the continuous state space of the chart statistic is partitioned into a finite number of discrete intervals and the probability distribution of the chart statistic is discretized to apply Markov chain approach. Let the interval of chart statistic Y_j which depends on $X_1, X_2, \cdots, X_j$ be divided into in-control region C and out-of-control region $C' = (h, \infty)$. Suppose that the region C is partitioned into r states $E_1, E_2, \cdots, E_r$ where each interval corresponds to a state of Markov chain and absorbing state $C' = \{x \mid Y_j > h\}$ is a signal region.

Since Y_j is continuous, let a discretized version $\tilde{Y}_j$ of $Y_j \in E_i$ be the midpoint of E_i. The probability of moving from any state i to any other state j can be denoted as $p_{ij}(k) = P(Y_{k+1} \in E_j \mid Y_k \in E_i)$ for $i, j = 1, 2, \cdots, r+1$ and $k = 0, 1, 2, \cdots$. In this paper, $p_{ij}(k)$ will be written briefly as p_{ij}. The transition probability matrix $P = [\, p_{ij}]$ can be partitioned as

$$P = \begin{bmatrix} Q & (I-Q) \cdot \underset{\sim}{1} \\ \underset{\sim}{1}' & 1 \end{bmatrix} \tag{4.1}$$

where Q is the $r \times r$ transition matrix corresponding to the transient state, I is the identity matrix, $\underset{\sim}{0}$ is an $r \times 1$ vector of 0's and $\underset{\sim}{1}$ is the $r \times 1$ vector of 1's. From the transition matrix P, we can obtain the fundamental matrix M as

$$M = (I - Q)^{-1} = [\, m_{ij}], \tag{4.2}$$

where m_{ij} is the expected number of visits to the transient state j before absorption, given that the Markov chain starts in transient state i.

For VSI chart, if we use a finite number of interval lengths $d_1, d_2, \cdots, d_\eta$ where $d_1 < d_2 < \cdots < d_\eta$. The continuous in-control region C be partitioned into η regions $C_1, C_2, \cdots, C_\eta$ where C_i is the region in which the interval d_i is used when $Y_j \in C_i$.

Let $\underset{\sim}{b} = (b_1, b_2, \cdots, b_r)$, $\underset{\sim}{N} = (N_1, N_2, \cdots, N_r)$ and $\underset{\sim}{T} = (T_1, T_2, \cdots, T_r)$ are vectors of sampling interval, NSS and TS, respectively. The ANSS vector is

$$E(\underset{\sim}{N}) = M\underset{\sim}{1} \tag{4.3}$$

and

$$V(\underset{\sim}{N}) = (2M - I) \cdot E(\underset{\sim}{N}) - [E(\underset{\sim}{N})]^{(2)}, \tag{4.4}$$

where $[E(\underset{\sim}{N})]^{(2)}$ is a vector whose ith component is the square of the ith component of $E(\underline{N})$. Hence when the process starts in state i, the ANSS N_i and the variancce $V(N_i)$ is given as

$$E(N_i) = \sum_{j=1}^{r} m_{ij} \tag{4.5}$$

and

$$V(N_i) = 2\sum_{k=1}^{r}\sum_{j=1}^{r} m_{ik}m_{kj} - \sum_{j=1}^{r} m_{ij} - (\sum_{j=1}^{r} m_{ij})^2 \tag{4.6}$$

Following Reynolds(1988), the ATS vector is

$$E(\underline{T}) = M\underline{b} \tag{4.7}$$

and

$$V(\underline{T}) = MB(2M-I)\underline{b} - [M\underline{b}]^{(2)}, \tag{4.8}$$

where B is a diagonal matrix with elements of corresponding sampling interval and $[M\underline{b}]^{(2)}$ is a vector whose ith component is the square of the ith component of $M\underline{b}$. Hence when the process starts in state i, the ATS T_i and the variancce $V(T_i)$ is given as

$$E(T_i) = \sum_{j=1}^{r} m_{ij}b_j \tag{4.9}$$

and

$$V(T_i) = 2\sum_{k=1}^{r}\sum_{j=1}^{r} m_{ik}m_{kj}b_k b_j - \sum_{j=1}^{r} m_{ij}b_j^2 - (\sum_{j=1}^{r} m_{ij}b_j)^2 \tag{4.10}$$

Let ATS(r) be an asymptotic ATS calculated using r states. Lucas and Crosier(1982) showed that an approximation of the continuous state ATS with a second degree polynomial in $1/r^2$ is a good approximation. This polynomial is of the form

$$ATS(r) = \textit{asymptotic}\ ATS + \frac{A}{r^2} + \frac{B}{r^4} \tag{4.11}$$

where A and B are the coefficients. For large r, we also obtain the ATS by taking the asymptotic ATS. The ATS(r) tends to stabilize as the number of transient state r increases.

In the following sections, we present $p_{ij}'s$ of transition probability matrix P in FSI chart and VSI with two sampling intervals, and we denote $F(\cdot)$ as the distribution function of control statistic. For the VSI feature, the sampling interval

d_1 is used when $Y_j \in C_1$ and the sampling interval d_2 is used when $Y_j \in C_2$. Then, the ATS when the Markov chain starts in state i is

$$ATS_i = d_2 \sum_{j=1}^{m} m_{ij} + d_1 \sum_{j=m+1}^{r} m_{ij}. \tag{4.12}$$

4.1 FSI EWMA Chart

A multivariate EWMA chart based on the statistic Z_i^2 $(i=1, 2, \cdots)$ is

$$Y_{Z^2, i} = (1-\lambda) Y_{Z^2, i-1} + \lambda Z_i^2 \tag{4.13}$$

where $Y_{Z^2, 0} = \omega \,(\omega \geq 0)$ and $0 < \lambda \leq 1$. This chart signals whenever $Y_{Z^2, i} \geq h$. Let the interval $(0, \infty)$ be divided into in-control region $C = (0, h]$ and out-of-control region $C' = (h, \infty)$. Assume that the in-control region is divided into r states then $\omega = h/r$.

Then the transition probability p_{ij} is as follows : For $i=1,2,\cdots,r$,

$$\begin{aligned} p_{ij} &= P[\,Y_k \in E_j \mid \text{in state } i] \\ &= F[\,(i-\tfrac{1}{2})w + (j-i+\tfrac{1}{2})w/\lambda] \\ &\qquad - F[\,(i-\tfrac{1}{2})w + (j-i-\tfrac{1}{2})w/\lambda], \qquad j=1,2,\cdots,r \\ p_{i,r+1} &= 1 - F[\left\{h-(1-\lambda)(i-\tfrac{1}{2})\,w\right\}/\lambda\,]. \end{aligned}$$

<Table 1> ANSS/ATS values for multivariate charts (p=3)

	Shewhart		EWMA					
shift	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
in-control	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
τ= 0.5	228.9	214.5	160.7	139.9	162.6	132.9	191.1	162.5
τ= 1.0	85.8	66.4	53.5	53.2	44.3	32.5	52.2	29.5
τ= 1.5	30.9	17.9	27.0	31.8	18.9	17.2	16.7	8.2
τ= 2.0	12.3	5.3	16.7	21.2	10.9	11.4	7.7	4.7
τ= 2.5	5.7	2.2	11.4	15.2	7.3	8.3	4.6	3.5
τ= 3.0	3.1	1.4	8.3	11.4	5.3	6.3	3.1	2.8
τ= 3.5	2.0	1.1	6.4	8.9	4.0	5.0	2.4	2.4
τ= 4.0	1.4	1.1	5.1	7.2	3.2	4.1	1.9	2.1
			$\lambda=0.05$		$\lambda=0.1$		$\lambda=0.3$	

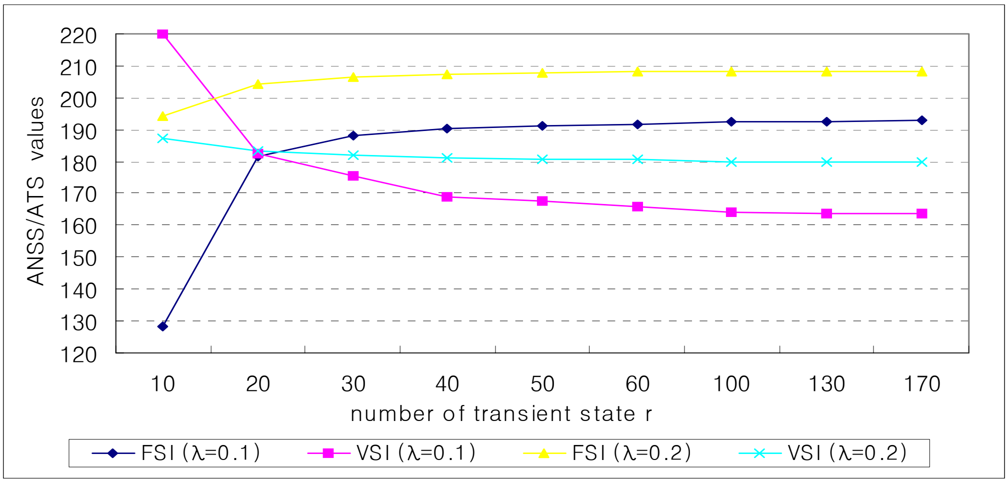
4.2 VSI EWMA Chart

For the two sampling intervals VSI EWMA chart based on Z_i^2 in (4.13), suppose the sampling interval ;

$$d_1 \text{ is used when } Y_{Z^2,i} \in (g, h]$$
$$d_2 \text{ is used when } Y_{Z^2,i} \in (0, g].$$

Suppose that this chart signals when $Y_{Z^2,i} \in C'$, the sampling interval d_2 is used when $Y_{Z^2,i} \in (g, h]$ and d_1 is used when $Y_{Z^2,i} \in (0, g]$. Assume that the interval $(0, g]$ is divided into m states and $(g, h]$ is divided into $(r-m)$ states then ω and v are g/m and $(h-g)/(r-m)$, respectively.

Then the transition probability p_{ij} is as follows : For $i=1, 2, \cdots, m$,



<Figure 1> Asymptotic ANSS/ATS with different number of r ($p=5$, $\tau^2=0.25$)

$$p_{ij} = \begin{cases} F\left[(i-\frac{1}{2})w + (j-i+\frac{1}{2})w/\lambda \right] - \\ \qquad F\left[(i-\frac{1}{2})w + (j-i-\frac{1}{2})w/\lambda \right], & j=1, \cdots, m \\ F\left[(i-\frac{1}{2}) + \left\{ g+(j-m)v-(i-\frac{1}{2})w \right\}/\lambda \right] \\ \qquad -F\left[(i-\frac{1}{2})w + \left\{ g+(j-m-1)v-(i-\frac{1}{2})w \right\}/\lambda \right], & j=m+1, \cdots r \\ 1-F\left[(i-\frac{1}{2})w + h-(i-\frac{1}{2})w/\lambda \right], & j=r+1. \end{cases}$$

For $i = m+1, m+2, \cdots, r,$

$$p_{ij} = \begin{cases} F\left[g + (i-m-\frac{1}{2})v + \left\{jw - g - (i-m-\frac{1}{2})v\right\}/\lambda\right] \\ \quad - F\left[g + (i-m-\frac{1}{2})v + \left\{(j-1)w - g - (i-m-\frac{1}{2})v\right\}/\lambda\right], & j = 1, \cdots, m \\ F\left[g + (i-m-\frac{1}{2})v + \left\{(j-i+\frac{1}{2})v\right\}/\lambda\right] \\ \quad - F\left[g + (i-m-\frac{1}{2})v + \left\{(j-m-\frac{1}{2})v\right\}/\lambda\right], & j = m+1, \cdots, r \\ 1 - F\left[g + (i-m-\frac{1}{2})v + \left\{h - g - (i-m-\frac{1}{2})v\right\}/\lambda\right], & j = r+1. \end{cases}$$

5. Numerical Results and Concluding Remarks

To evaluate the performances and compare for matched FSI and VSI multivariate EWMA charts with Markov chain approach, we let that the sampling interval of unit time $d = 1$ in FSI charts and two sampling intervals $d_1 = 0.1$ and $d_2 = 1.9$ for the VSI charts. In our computation, the ANSS and ATS of the chart when the process is in-control were fixed to be 370.4 and the sample size for each characteristic was 5 for p=3, 5.

The parameters h and g values of Shewhart chart in (3.1) are obtained from the percentage points of chi-square distributions to satisfy an in-control ATS and ANSS. After the smoothing constant of the proposed EWMA chart in (4.13) has been determined, the parameters h and g were caluculated by Markov chain method.

The ATS for matched FSI and VSI charts are given in <Table 1> and <Table 2>. Numerical computation shows that smaller values of smoothing constant λ is efficient for small shift and VSI feature are more efficient. These results coincides with univariate EWMA chart. And the asymptotic ANSS and ATS values with different number of transient state r are given in <Figure 1>. When r is small, for example r=10～60, the asymptotic numerical values are not stabilize. In our computation, ATS(r) tends to be stabilize when the number r is greater than 100 for various p.

<Table 2> ANSS/ATS values for multivariate charts (p=5)

shift	Shewhart		EWMA					
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
in-control	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
τ= 0.5	259.4	246.1	190.3	169.9	192.5	164.3	221.0	194.8
τ= 1.0	114.1	92.7	69.8	69.0	59.3	44.4	71.1	44.7
τ= 1.5	44.4	28.0	36.3	43.2	25.4	23.3	23.4	11.9
τ= 2.0	17.9	8.4	23.0	30.1	14.8	15.9	10.4	6.3
τ= 2.5	8.1	3.1	16.1	22.1	9.9	11.7	6.0	4.5
τ= 3.0	4.2	1.6	11.9	16.9	7.2	9.0	4.0	3.6
τ= 3.5	2.5	1.2	9.2	13.3	5.6	7.2	3.0	3.0
τ= 4.0	1.7	1.1	7.3	10.7	4.4	5.9	2.4	2.5
			$\lambda=0.05$		$\lambda=0.1$		$\lambda=0.3$	

References

1. Amin, R.W. and Lestinger, W.C.(1991), "Improved Switching Rules in Control Procedures Using Variable Sampling Intervals," *Communications in Statistics – Simmulation and Computation*, Vol.20, pp.205-230.
2. Brook, D. and Evans, D.A.(1972), "An Approach to the Probability Distribution of CUSUM Run Length," *Biometrika*, Vol.59, pp.539-549.
3. Cui, R.Q. and Reynolds, M.R.,Jr.(1988), "$\overline{X}$-charts with Runs Rules and Variable Sampling Intervals," *Communications in Statistics - Simulation and Computations*, Vol.17, pp.1073-1093.
4. Lucas, J.M. and Saccucci, M.S.(1990), "Exponentially Weighted Moving Average Control Schemes : Properties and Enhancements," *Technometrics*, Vol.32, pp.1-12.
5. Reynolds, M.R., Jr.(1988), "Markovian Variable Sampling Interval Control Charts" Technical Report 88-22, Virginia Polytechnic Institute and State University, Dept. of Statistics.
6. Reynolds, M.R., Jr.(1995), "Optimal Variable Sampling Interval Control Charts with Variable Sampling Intervals," *Sequential Analysis*, Vol.8, pp.361-379.
7. Woodall, W.H.(1983), "The Distribution of the Run Length of One-Sided CUSUM Procedures for Continuous Random Variables," *Technometrics*, Vol.25, pp.295-301.
8. Woodall, W.H. and Ncube, M.M.(1985), "Multivariate CUSUM Quality Control Procedure," *Technometrics*, Vol.27, pp.285-292.

[received date : Sep. 2003, accepted date : Nov. 2003]