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Recurrence Relations in the Transformed Exponential Distributions

Jeen Kap Choi[1] · Kap Jong Mo[2]

Abstract

In this paper, we establish some recurrence relations of the moments, product moments, percentage points, and modes of order statistics from the transformed exponential distribution.

Keywords: Recurrence relation, Order statistics, Transformed exponential distribution

1. Introduction

We consider recurrence relations in the transformed exponential distribution. Let the random variable Y have a cumulative distribution function

$$G(t) \equiv 1 - \frac{2}{1+e^{\frac{t}{\mu}}}, \quad 0 \le t < \infty,$$

and a probability density function(pdf)

$$g(t) = \frac{2e^{\frac{t}{\mu}}}{\mu(1+e^{\frac{t}{\mu}})^2}, \quad 0 \le t < \infty. \tag{1.1}$$

Note that $g(t)$ and $G(t)$ satisfy the relations

$$g(t) = \frac{1}{\mu} G(t)\{1-G(t)\} + \frac{1}{2\mu}\{1-G(t)\}^2, \tag{1.2}$$

1) Professor, Department of Statistics, Kyungpook National University, Daegu, 702-701, Korea
E-Mail : choijk@knu.ac.kr
2) Part-time Lecturer, Department of Statistics, Kyungpook National University, Daegu, 702-701, Korea

$$g(t)=\frac{1}{\mu}\{1-G(t)\}-\frac{1}{2\mu}\{1-G(t)\}^2, \qquad (1.3)$$

and

$$g(t)=\frac{1}{2\mu}\{1-G^2(t)\}. \qquad (1.4)$$

Let $Y_1, Y_2, \cdots, Y_n$ be a random sample of size n from (1.1), and let $Y_{1:n}\le Y_{2:n}\le\cdots\le Y_{n:n}$ be the corresponding order statistics. Denote the k th moment $E(Y^k_{r:n})$ by $\alpha^{(k)}_{r:n}$ $(1\le r\le n; k\ge 0)$, the product moment $E(Y_{r:n}Y_{s:n})$ by $\alpha_{r,s:n}$ $(1\le r< s\le n)$, the $100p$-percentage points of $Y_{r:n}$ $(1\le r\le n)$ by $\xi_{p,r:n}$, the mode of $Y_{r:n}$ by $m_{r:n}$ and the covariance between $Y_{r:n}$ and $Y_{s:n}$ by $\sigma_{r,s:n}$, $(1\le r< s\le n)$. For convenience we will use $\alpha_{r:n}$ for $\alpha^{(1)}_{r:n}$, $\alpha_{r,r:n}$ for $\alpha^{(2)}_{r:n}$, and $\sigma_{r,r:n}$ for variance of $Y_{r:n}$, $1\le r\le n$.

Order statistics and their moments are of great importance in many statistical problems. Linear functions of the order statistics are extremely useful in the estimation of parameters and also in testing hypotheses problems. The application of Gauss-Markov theorem of least squares by Lloyd (1952) (also see Sarhan and Greenberg, 1962) to derive linear functions of order statistics (termed as linear estimators) for estimating the parameters of distributions depending on location and scale, is one fine example. Knowledge of the moments of order statistics, in particular their means, variances and covariances, allows us to find the expected value and variance of the linear function, and hence permits us to obtain estimators and their efficiencies.

For the logistic distribution, moments of order statistics have been studied in great detail by several authors, for example, see Birnbaum and Dudman (1963), Gupta and Shah (1965), Tarter and Clark (1965), Shah (1966, 1970), and Gupta, Qureishi and Shah (1967). In particular, Gupta and Shah (1965) has given exact moments and percentage points of the order statistics, and Shah (1966, 1970) has obtained many recurrence relations for the moments and product moments of order statistics. Tarter (1966) has given explicit finite series expressions for the moments of order statistics from a doubly truncated logistic distribution. Recently, Balakrishnan and Joshi (1983a, b) have obtained several recurrence relations for the moments and product moments of order statistics from a symmetrically truncated logistic distribution and applied them to tabulate the means, variances and covariances. In particular, Balakrishnan(1985) has obtained several recurrence relations of the moments and product moments of order statistics from the half logistic distribution and have given exact moments and percentage points of the order statistics.

In this paper, we establish similar recurrence relations of the moments, product moments and also obtain percentage points and modes of order statistics from the transformed exponential distribution. The importance of the logistic and loglogistic distributions in survival analysis is well known (for example, see Cox, 1970; Bennett, 1983). Plotting the survival functions is a useful tool in the analysis of fitting medical data (Gross and Clark, 1975). Analogous to the applications of the half normal distribution in plotting the residues in regression analysis, therefore, one may find some applications of the transformed exponential distribution in plotting the survival functions. These problems need further investigation and are currently being studied.

2. Recurrence relations for moments

With $g(y)$ as given in (1.1), the pdf of $Y_{r:n}$, $1 \le r \le n$, is

$$f_{r:n}(y) = C_{r,n}\{G(y)\}^{r-1}\{1-G(y)\}^{n-r}g(y),\ 0 \le y < \infty, \tag{2.1}$$

where $C_{r,n} = B(r, n-r+1)^{-1}$, $B(a,b)$ being the beta function given by

$$B(a,b) \equiv \Gamma(a)\Gamma(b)/\Gamma(a+b), \qquad a, b > 0.$$

Then the k th moments $\alpha_{r:n}^{(k)}$ of $Y_{r:n}$ satisfy the following relations.

Theorem 2.1. For $n \ge 1$ and $k = 0, 1, 2, \cdots$,

$$\alpha_{1:n+1}^{(k+1)} = 2\left[\alpha_{1:n}^{(k+1)} - \left\{\frac{\mu(k+1)}{n}\right\}\alpha_{1:n}^{(k)}\right]$$

with the convention $\alpha_{1:n}^{(0)} = 1$. (2.2)

proof.

For $n \ge 1$ we have from (2.1)

$$\begin{aligned}\alpha_{1:n}^{(k)} &= E(Y_{1:n}^k)\\ &= n\int_0^\infty y^k\{1-G(y)\}^{n-1}g(y)dy\\ &= n\int_0^\infty y^k\{1-G(y)\}^{n-1}\left[\frac{1}{\mu}\{1-G(y)\} - \frac{1}{2\mu}\{1-G(y)\}^2\right]dy\\ &= \frac{n}{\mu}\left[\int_0^\infty y^k\{1-G(y)\}^n dy - \frac{1}{2}\int_0^\infty y^k\{1-G(y)\}^{n+1}dy\right].\end{aligned}$$

Now integrating by parts treating y^k for integration and the rest of the integrand for differentiation and simplifying the resulting expression, we get (2.2). ■

Theorem 2.2. For $1\le r\le n$ and $k=0,1,2,\cdots$,

$$\alpha^{(k+1)}_{r+1:n+1}=\frac{1}{r}\left[\left\{\frac{\mu(n+1)(k+1)}{n-r+1}\right\}\alpha^{(k)}_{r:n}+\frac{n+1}{2}\alpha^{(k+1)}_{r-1:n}-\frac{n-2r+1}{2}\alpha^{(k+1)}_{r:n+1}\right] \tag{2.3}$$

with the conventions that $\alpha^{(k)}_{0:t}=0$ for $t\ge 1$ and $k=0,1,2,\cdots$, and $\alpha^{(0)}_{r:t}=1$ for $1\le r\le t$.

proof.

Using (1.2), we have for $1\le r\le n$

$$\begin{aligned}\alpha^{(k)}_{r:n} &= E(Y^k_{r:n})\\ &= C_{r,n}\int_0^\infty y^k\{G(y)\}^{r-1}\{1-G(y)\}^{n-r}g(y)dy\\ &= \frac{C_{r,n}}{\mu}\int_0^\infty y^k\{G(y)\}^{r-1}\{1-G(y)\}^{n-r}\left[G(y)\{1-G(y)\}+\frac{1}{2}\{1-G(y)\}^2\right]dy\\ &= \frac{C_{r,n}}{\mu}(I_1+\frac{1}{2}I_2),\end{aligned} \tag{2.4}$$

where

$$I_1=\int_0^\infty y^k\{G(y)\}^{r}\{1-G(y)\}^{n-r+1}dy$$

and

$$I_2=\int_0^\infty y^k\{G(y)\}^{r-1}\{1-G(y)\}^{n-r+2}dy.$$

Integrating by parts treating y^k for integration and the rest of the integrand for differentiation, we get I_1

$$\begin{aligned}I_1 &= \frac{1}{k+1}\left[(n-r+1)\int_0^\infty y^{k+1}\{G(y)\}^{r}\{1-G(y)\}^{n-r}g(y)dy\right.\\ &\qquad \left.-r\int_0^\infty y^{k+1}\{G(y)\}^{r-1}\{1-G(y)\}^{n-r+1}g(y)dy\right]\\ &= \frac{1}{k+1}\{(n-r+1)C^{-1}_{r+1,n+1}\,\alpha^{(k+1)}_{r+1:n+1}-rC^{-1}_{r,n+1}\,\alpha^{(k+1)}_{r:n+1}\}\\ &= \frac{1}{k+1}C^{-1}_{r+1,n+2}\{\alpha^{(k+1)}_{r+1:n+1}-\alpha^{(k+1)}_{r:n+1}\}(n+2).\end{aligned}$$

By changing r to $r-1$ in the above expression, we can get

$$I_2=\frac{1}{k+1}C^{-1}_{r,n+2}\{\alpha^{(k+1)}_{r:n+1}-\alpha^{(k+1)}_{r-1:n+1}\}(n+2).$$

Note that this expression holds good for the case $r=1$ as well with the convention that $\alpha^{(k)}_{0:t}=0$ for $t\ge 1$ and $k=1,2,\cdots$. Substituting the above expressions for I_1 and I_2 into (2.4) and simplifying, we get

$$\alpha_{r:n}^{(k)}=\frac{n-r+1}{\mu(n+1)(k+1)}\left[r\{\alpha_{r+1:n+1}^{(k+1)}-\alpha_{r:n+1}^{(k+1)}\}+\frac{n-r+2}{2}\{\alpha_{r:n+1}^{(k+1)}-\alpha_{r-1:n+1}^{(k+1)}\}\right]. \tag{2.5}$$

Also, using the well-known recurrence relation

$$r\alpha_{r+1:n}^{(k+1)}+(n-r)\alpha_{r:n}^{(k+1)}=n\alpha_{r:n-1}^{(k+1)},$$

we have

$$(n-r+2)\{\alpha_{r:n+1}^{(k+1)}-\alpha_{r-1:n+1}^{(k+1)}\}=(n+1)\{\alpha_{r:n+1}^{(k+1)}-\alpha_{r-1:n}^{(k+1)}\}.$$

Finally, substituting this expression into (2.5) and simplifying, we get the relation given at (2.3) ■

3. Recurrence relations for product moments

The joint pdf of $Y_{r:n}$ and $Y_{s:n}$ $(1\le r<s\le n)$ is given by

$$f_{r,s:n}(x,y)=C_{r,s,n}\{G(x)\}^{r-1}\{G(y)-G(x)\}^{s-r-1}\{1-G(y)\}^{n-s}g(x)g(y), \tag{3.1}$$

$0\le x<y<\infty$,

where

$$C_{r,s,n}=B(r,s-r,n-s+1)^{-1}, \tag{3.2}$$

where $B(a,b,c)$ is the generalized beta function defined by $B(a,b,c)\equiv\Gamma(a)\Gamma(b)\Gamma(c)/\Gamma(a+b+c)$, $(a,b,c>0)$, with $g(x)$ as in (1.1). The product moments $\alpha_{r,s:n}$ satisfy the following relations.

Theorem 3.1. For $1\le r\le n-1$, we have

$$\alpha_{r,r+1:n+1}=\alpha_{r:n+1}^{(2)}+\frac{2(n+1)}{n-r+1}\left\{\alpha_{r,r+1:n}-\alpha_{r:n}^{(2)}-\frac{\mu\alpha_{r:n}}{n-r}\right\}. \tag{3.3}$$

proof.

For $1\le r\le n-1$, write

$$\begin{aligned}\alpha_{r:n}&=E(Y_{r:n}Y^{0}_{r+1:n})\\&=\int_0^\infty\int_x^\infty xC_{r,r+1,n}\{G(x)\}^{r-1}\{G(y)-G(x)\}^{0}\{1-G(y)\}^{n-r-1}g(x)g(y)dydx\\&=C_{r,r+1,n}\int_0^\infty\int_x^\infty x\{G(x)\}^{r-1}\{1-G(y)\}^{n-r-1}g(x)g(y)dydx\\&=C_{r,r+1,n}\int_0^\infty x\{G(x)\}^{r-1}g(x)\left[\int_x^\infty\{1-G(y)\}^{n-r-1}g(y)dy\right]dx,\end{aligned}$$

where $C_{r,r+1,n}$ is as in (3.2). Now consider

(3.4)

$$D= \int_x^{\infty}\{1-G(y)\}^{n-r-1}g(y)dy$$

$$= \frac{1}{\mu}\int_x^{\infty}\{1-G(y)\}^{n-r}dy-\frac{1}{2\mu}\int_x^{\infty}\{1-G(y)\}^{n-r+1}dy,$$

using (1.3). Integrating by parts now, we get

$$D= \frac{1}{\mu}\Big[(n-r)\int_x^{\infty}y\{1-G(y)\}^{n-r-1}g(y)dy-x\{1-G(x)\}^{n-r}$$

$$+\frac{x}{2}\{1-G(x)\}^{n-r+1}-\frac{n-r+1}{2}\int_x^{\infty}y\{1-G(y)\}^{n-r}g(y)dy\Big].$$

Finally, substituting this expression into (3.4) and simplifying the resulting expression, the relation at (3.3) follows immediately.

Theorem 3.2. For $n\geq 2$,

$$\alpha_{2,3:n+1}=\alpha^{(2)}_{3:n+1}+(n+1)\Big\{\mu\alpha_{2:n}-\frac{n}{2}\alpha^{(2)}_{1:n-1}\Big\} \tag{3.5}$$

and for $2\leq r\leq n-1$,

$$\alpha_{r+1,r+2:n+1}=\alpha^{(2)}_{r+2:n+1}+\frac{n+1}{r(r+1)}\{2\mu\alpha_{r+1:n}+n(\alpha_{r-1,r:n-1}-\alpha^{(2)}_{r:n-1})\}. \tag{3.6}$$

proof.

For $1\leq r\leq n-1$,

$$\alpha_{r+1:n}= E(Y^0_{r:n}Y_{r+1:n})$$

$$= C_{r,r+1,n}\int_0^{\infty}y\{1-G(y)\}^{n-r-1}g(y)\Big[\int_0^y\{G(x)\}^{r-1}g(x)dx\Big]dy$$

$$= C_{r,r+1,n}\int_0^{\infty}y\{1-G(y)\}^{n-r-1}g(y)$$

$$\times\frac{1}{2\mu}\Big[\int_0^y\{G(x)\}^{r-1}dx-\int_0^y\{G(x)\}^{r+1}dx\Big]dy, \tag{3.7}$$

where $C_{r,r+1,n}$ is defined in (3.2).

i) $r=1$

$$\alpha_{2:n} = C_{1,2,n}\int_0^{\infty}y\{1-G(y)\}^{n-2}g(y)\times\frac{1}{2\mu}\Big[\int_0^y dx-\int_0^y\{G(x)\}^2dx\Big]dy$$

$$=\frac{1}{2\mu}\Big(n\alpha^{(2)}_{1:n-1}-\frac{2}{n+1}\alpha^{(2)}_{3:n+1}+\frac{2}{n+1}\alpha_{2,3:n+1}\Big),$$

therefore $\alpha_{2,3:n+1}=\alpha^{(2)}_{3:n+1}+(n+1)\Big(\mu\alpha_{2:n}-\frac{n}{2}\alpha^{(2)}_{1:n-1}\Big).$

ii) $2\le r\le n-1$

$$\alpha_{r+1:n}= C_{r,r+1,n}\int_0^\infty y\{1-G(y)\}^{n-r-1}g(y)$$
$$\times\frac{1}{2\mu}\Big[y\{G(y)\}^{r-1}-(r-1)\int_0^y x\{G(x)\}^{r-2}g(x)dx-y\{G(y)\}^{r+1}$$
$$+(r+1)\int_0^y x\{G(x)\}^{r}g(x)dx\Big]dy$$

hence

$$\alpha_{r+1,r+2:n+1}=\alpha^{(2)}_{r+2:n+1}+\frac{n+1}{r(r+1)}\{2\mu\alpha_{r+1:n}+n(\alpha_{r-1,r:n-1}-\alpha^{(2)}_{r:n-1})\}. \quad ■$$

In particular, setting $r=n-1$ in (3.6), we get the relation

$$\alpha_{n,n+1:n+1}=\alpha^{(2)}_{n+1:n+1}+\frac{n+1}{(n-1)n}\{2\mu\alpha_{n:n}+n(\alpha_{n-2,n-1:n-1}-\alpha^{(2)}_{n-1:n-1})\} \quad (3.8)$$

for $n\ge 3$. Also for any arbitrary continuous distribution, we have

$$\alpha_{1,2:2}=\alpha^2_{1:1}$$

(Govindarajulu, 1963). Along with this, the relations in (3.3), (3.5) and (3.6) give the moments $\alpha_{r,r+1:n}(1\le r\le n-1)$. After computing $\alpha_{1,2:2}$, $\alpha_{1,2:3}$ and $\alpha_{2,3:3}$ could be computed using (3.3) and (3.5) respectively. Next, $\alpha_{1,2:4}$ and $\alpha_{2,3:4}$ could be computed by (3.3) and (3.8) could be used to evaluate $\alpha_{3,4:4}$. Thus in a systematic recursive way, for a sample of size n, the product moments $\alpha_{r,r+1:n}(1\le r\le n-1)$ can all be calculated. Note that this is sufficient for the evaluation of all the product moments $\alpha_{r,s:n}$ as the remaining moments, $\alpha_{r,s:n}(s-r\ge 2)$ can all be computed by using the well-known recurrence relation:

$$(r-1)\alpha_{r,s:n}+(s-r)\alpha_{r-1,s:n}+(n-s+1)\alpha_{r-1,s-1:n}=n\alpha_{r-1,s-1:n-1}. \quad (3.9)$$

However, for the sake of completeness, we establish some more recurrence relations satisfied by the product moments $\alpha_{r,s:n}(s-r\ge 2)$, which can be proved by following an exactly similar approach.

Theorem 3.3. For $1\le r\le n-2$ and $s-r\ge 2$,

$$\alpha_{r,s:n+1}=\alpha_{r,s-1:n+1}+\frac{2(n+1)}{n-s+2}\left\{\alpha_{r,s:n}-\alpha_{r,s-1:n}-\frac{\mu\alpha_{r:n}}{n-s+1}\right\}.$$

proof.

For $1\le r\le n-2$,

$$\alpha_{r:n} = E(Y_{r:n} Y^{0}_{s:n})$$
$$= C_{r,s,n} \int_0^\infty \int_x^\infty x\{G(x)\}^{r-1}\{G(y)-G(x)\}^{s-r-1}\{1-G(y)\}^{n-s} g(x)g(y)dxdy$$
$$= C_{r,s,n} \int_0^\infty x\{G(x)\}^{r-1} g(x)\Big[\int_x^\infty \{G(y)-G(x)\}^{s-r-1}\{1-G(y)\}^{n-s} g(y)dy\Big]dx,$$

where $C_{r,s,n}$ is defined in (3.2). Now consider

$$D = \int_x^\infty \{G(y)-G(x)\}^{s-r-1}\{1-G(y)\}^{n-s} g(y)dy$$
$$= \frac{1}{\mu}\int_x^\infty \{G(y)-G(x)\}^{s-r-1}\{1-G(y)\}^{n-s+1}dy$$
$$-\frac{1}{2\mu}\int_x^\infty \{G(y)-G(x)\}^{s-r-1}\{1-G(y)\}^{n-s+2}dy$$
$$= \frac{1}{\mu}\Big[(n-s+1)\int_x^\infty y\{G(y)-G(x)\}^{s-r-1}\{1-G(y)\}^{n-s} g(y)dy$$
$$-(s-r-1)\int_x^\infty y\{G(y)-G(x)\}^{s-r-2}\{1-G(y)\}^{n-s+1} g(y)dy\Big]$$
$$-\frac{1}{2\mu}\Big[(n-s+2)\int_x^\infty y\{G(y)-G(x)\}^{s-r-1}\{1-G(y)\}^{n-s+1} g(y)dy$$
$$-(s-r-1)\int_x^\infty y\{G(y)-G(x)\}^{s-r-2}\{1-G(y)\}^{n-s+2} g(y)dy\Big].$$

Therefore

$$\alpha_{r:n} = \frac{1}{\mu}\Big[\int_0^\infty \int_x^\infty (n-s+1)C_{r,s,n} xy\{G(x)\}^{r-1}\{G(y)-G(x)\}^{s-r-1}$$
$$\{1-G(y)\}^{n-s} g(x)g(y)dydx - \int_0^\infty \int_x^\infty (s-r-1)C_{r,s,n} xy$$
$$\{G(x)\}^{r-1}\{G(y)-G(x)\}^{s-r-2}\{1-G(y)\}^{n-s+1} g(x)g(y)dydx]$$
$$-\frac{1}{2\mu}\Big[\int_0^\infty \int_x^\infty (n-s+2)C_{r,s,n} xy\{G(x)\}^{r-1}\{G(y)-G(x)\}^{s-r-1}$$
$$\{1-G(y)\}^{n-s+1} g(x)g(y)dydx - \int_0^\infty \int_x^\infty (s-r-1)C_{r,s,n} xy$$
$$\{G(x)\}^{r-1}\{G(y)-G(x)\}^{s-r-2}\{1-G(y)\}^{n-s+2} g(x)g(y)dydx]$$
$$= \frac{1}{\mu}\{(n-s+1)\alpha_{r,s:n} - (n-s+1)\alpha_{r,s-1:n}\}$$
$$-\frac{1}{2\mu}\Big\{\frac{(n-s+1)(n-s+2)}{n+1}\alpha_{r,s:n+1} - \frac{(n-s+1)(n-s+2)}{n+1}\alpha_{r,s-1:n+1}\Big\}.$$

Hence

$$\alpha_{r,s:n+1}=\alpha_{r,s-1:n+1}+\frac{2(n+1)}{n-s+2}\left\{\alpha_{r,s:n}-\alpha_{r,s-1:n}-\frac{\mu\alpha_{r:n}}{n-s+1}\right\}. \qquad ■$$

Theorem 3.4. For $3\leq s\leq n$,

$$\alpha_{2,s+1:n+1}=\alpha_{3,s+1:n+1}+(n+1)\left\{\mu\alpha_{s:n}-\frac{n}{2}\alpha_{1,s-1:n-1}\right\}$$

and for $2\leq r\leq n-2$ and $s-r\geq 2$,

$$\alpha_{r+1,s+1:n+1}=\alpha_{r+2,s+1:n+1}+\frac{n+1}{r(r+1)}\{2\mu\alpha_{s:n}-n(\alpha_{r,s-1:n-1}-\alpha_{r-1,s-1:n-1})\}.$$

proof.

For $3\leq s\leq n$,

$$\alpha_{s:n}=E(Y^0_{r:n}Y_{s:n})$$

$$=C_{r,s,n}\int_0^{\infty}\int_0^{y}y\{G(x)\}^{r-1}\{G(y)-G(x)\}^{s-r-1}\{1-G(y)\}^{n-s}g(x)g(y)dxdy$$

$$=C_{r,s,n}\int_0^{\infty}y\{1-G(y)\}^{n-s}g(y)\left[\int_0^{y}\{G(x)\}^{r-1}\{G(y)-G(x)\}^{s-r-1}g(x)dx\right]dy$$

$$=C_{r,s,n}\int_0^{\infty}y\{1-G(y)\}^{n-s}g(y)\frac{1}{2\mu}\left[\int_0^{y}\{G(x)\}^{r-1}\{G(y)-G(x)\}^{s-r-1}dx\right.$$

$$\left.-\int_0^{y}\{G(x)\}^{r+1}\{G(y)-G(x)\}^{s-r-1}dx\right]dy.$$

i) $r=1$

$$\alpha_{s:n}=\frac{1}{2\mu}\int_0^{\infty}C_{1,s,n}y\{1-G(y)\}^{n-s}g(y)\left[\int_0^{y}\{G(y)-G(x)\}^{s-2}dx\right.$$

$$\left.-\int_0^{y}\{G(x)\}^{2}\{G(y)-G(x)\}^{s-2}dx\right]dy$$

$$=\frac{1}{2\mu}\int_0^{\infty}C_{1,s,n}y\{1-G(y)\}^{n-s}g(y)\left[(s-2)\int_0^{y}x\{G(y)-G(x)\}^{s-3}g(x)dx\right.$$

$$+2\int_0^{y}xG(x)\{G(y)-G(x)\}^{s-2}g(x)dx$$

$$\left.-(s-2)\int_0^{y}x\{G(x)\}^{2}\{G(y)-G(x)\}^{s-3}g(x)dx\right]dy$$

$$=\frac{1}{2\mu}\int_0^{\infty}\int_0^{y}(s-2)C_{1,s,n}xy\{G(y)-G(x)\}^{s-3}\{1-G(y)\}^{n-s}g(x)g(y)dxdy$$

$$+\frac{1}{\mu}\int_0^{\infty}\int_0^{y}C_{1,s,n}xyG(x)\{G(y)-G(x)\}^{s-2}\{1-G(y)\}^{n-s}g(x)g(y)dxdy$$

$$-\frac{1}{2\mu}\int_0^{\infty}\int_0^{y}(s-2)C_{1,s,n}xy\{G(x)\}^{2}\{G(y)-G(x)\}^{s-3}\{1-G(y)\}^{n-s}g(x)g(y)dxdy.$$

Thus

$$\mu\alpha_{s:n}=\frac{1}{n+1}(\alpha_{2,s+1:n+1}-\alpha_{3,s+1:n+1})+\frac{n}{2}\alpha_{1,s-1:n-1}$$

and hence $\alpha_{2,s+1:n+1}=\alpha_{3,s+1:n+1}+(n+1)\left\{\mu\alpha_{s:n}-\frac{n}{2}\alpha_{1,s-1:n-1}\right\}$.

ii) $2\leq r\leq n-2$

$$\begin{aligned}\alpha_{s:n}= &\frac{1}{2\mu}\int_0^\infty\int_0^y(s-r-1)C_{r,s,n}xy\{G(x)\}^{r-1}\{G(y)-G(x)\}^{s-r-2}\\ &\times\{1-G(y)\}^{n-s}g(x)g(y)dxdy-\frac{1}{2\mu}\int_0^\infty\int_0^y(r-1)C_{r,s,n}xy\{G(x)\}^{r-2}\\ &\times\{G(y)-G(x)\}^{s-r-1}\{1-G(y)\}^{n-s}g(x)g(y)dxdy\\ &+\frac{1}{2\mu}\int_0^\infty\int_0^y(r+1)C_{r,s,n}xy\{G(x)\}^{r}\{G(y)-G(x)\}^{s-r-1}\\ &\times\{1-G(y)\}^{n-s}g(x)g(y)dxdy\\ &-\frac{1}{2\mu}\int_0^\infty\int_0^y(s-r-1)C_{r,s,n}xy\{G(x)\}^{r+1}\{G(y)-G(x)\}^{s-r-2}\\ &\times\{1-G(y)\}^{n-s}g(x)g(y)dxdy.\end{aligned}$$

Therefore

$$\alpha_{r+1,s+1:n+1}=\alpha_{r+2,s+1:n+1}+\frac{n+1}{r(r+1)}\{2\mu\alpha_{s:n}-n(\alpha_{r,s-1:n-1}-\alpha_{r-1,s-1:n-1})\}. \qquad\blacksquare$$

4. Computations of means, variances, and covariances

The recurrence relations derived in sections 2 and 3 allow us to compute the means, variances and covariances of order statistics. Starting with $\alpha_{1:1}=\ln 4$ and $\alpha_{1:1}^{(2)}=\frac{\pi^2}{3}$, relations in (2.2) and (2.3) were used to compute the first two moments of all order statistics. These moments were calculated to 10 significant digits, and were checked by using the identities

$$\sum_{r=1}^{n}\alpha_{r:n}^{(k)}=n\alpha_{1:1}^{(k)},\ k=1,2.$$

For variances and covariances, the product moments $\alpha_{r,s:n}$ were all obtained in a systematic manner. First, the diagonal elements $\alpha_{r,r:n}(=\alpha_{r:n}^{(2)})$ were filled up. Then starting from $\alpha_{1,2:2}=\alpha_{1:1}^2$, the relations at (3.3), (3.5) and (3.8) were used to evaluate the elements just above the diagonal, viz., $\alpha_{r,r+1:n}(1\leq r\leq n-1)$. After that, $\alpha_{r,r+2:n}$ were calculated by using the recurrence relation given at (3.9)

and so on. Finally, $\alpha_{1,n:n}$ was calculated by using (3.9). The variance-covariance matrix $((\sigma_{r,s:n}))$, where
$\sigma_{r,s:n}=\alpha_{r,s:n}-\alpha_{r:n}\alpha_{s:n}$, was then computed to 10 significant digits. These were checked for their accuracy by the known identity

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\sigma_{r,s:n}=n\sigma_{1,1:1}$$

and the identities

$$\sum_{s=r+1}^{n}\sigma_{r,s:n}+\sum_{i=1}^{r}\sigma_{i,r+1:n}=\left(r\alpha_{1:1}-\sum_{i=1}^{r}\alpha_{i:n}\right)(\alpha_{r+1:n}-\alpha_{r:n})$$

which have been recently established by Joshi and Balakrishnan(1982) for an arbitrary continuous distribution.

5. Percentage points and modes of order statistics

The cumulative distribution function of $Y_{r:n}$ is given by

$$F_{r:n}(y)\equiv I_{G(y)}(r,n-r+1),\ 1\le r\le n,$$

where $I_\alpha(a,b)$ is the regularized incomplete beta function defined by

$$I_\alpha(a,b)\equiv\{B(a,b)\}^{-1}\int_0^\alpha t^{a-1}(1-t)^{b-1}dt,\ a,b>0.$$

Therefore, the $100p$-percentage points of $Y_{r:n}\,(1\le r\le n)$ $\xi_{p,r:n}$ can be obtained by solving the equation

$$I_{G(\xi_{p,r:n})}(r,n-r+1)=p. \tag{5.1}$$

Now note that the percentage points can be calculated from (5.1) either by using the tables of regularized incomplete beta function prepared by Pearson (1934), or by using the algorithm given by Cran *et al.* (1977). However, from (5.1), one can obtain an exact and explicit expression for the $100p$-percentage point of the smallest order statistic as

$$\xi_{p,1:n}=\mu\ln\left\{2(1-p)^{-\frac{1}{n}}-1\right\}$$

and of the largest order statistic as

$$\xi_{p,n:n}=\mu\ln\left\{(1+p^{\frac{1}{n}})/(1-p^{\frac{1}{n}})\right\}.$$

For $r=1$, (2.1) reduces to

$$f_{1:n}(y)=n\{1-G(y)\}^{n-1}g(y)$$
$$=\frac{n}{\mu}\left[\{1-G(y)\}^{n}-\frac{1}{2}\{1-G(y)\}^{n+1}\right],\ 0\le y<\infty,$$

using (1.3). Now for $n \geq 2$, we see that the derivative of $f_{1:n}(y)$ is less than zero which implies that $f_{1:n}(y)$ is a monotonically decreasing function of y in $[0, \infty)$. Consequently, $m_{1:n} = 0$ gives the mode of $Y_{1:n}$. Next for $r = n$, (2.1) gives

$$f_{n:n}(y) = n\{G(y)\}^{n-1} g(y),$$

$$= \frac{n}{2\mu}[\{G(y)\}^{n-1} - \{G(y)\}^{n+1}],\ 0 \leq y < \infty,$$

using (1.4). Differentiating and equating to zero, we get the mode of $X_{n:n}$, for $n \geq 1$, as

$$m_{n:n} = \mu \ln\{(1+h)/(1-h)\},$$

where $h = \{(n-1)/(n+1)\}^{\frac{1}{2}}$. Finally, for $2 \leq r \leq n-1$, (2.1) on using (1.2) gives

$$f_{r:n}(y) = \frac{C_{r,n}}{\mu}[\{G(y)\}^{r}\{1-G(y)\}^{n-r+1}$$

$$+ \frac{1}{2}\{G(y)\}^{r-1}\{1-G(y)\}^{n-r+2}],\ 0 \leq y < \infty.$$

Upon differentiating and equating to zero, we have

$$(n+1)G^2(y) + (n-r)G(y) - (r-1) = 0.$$

This is a quadratic equation in $G(y)$, with solutions

$$G(y) = \frac{1}{2(n+1)}[-(n-r) \pm \{(n-r)^2 + 4(r-1)(n+1)\}^{\frac{1}{2}}]. \tag{5.2}$$

Clearly one of these is negative. Further, starting with the inequality

$$\frac{r-1}{n-r} < 1 + \frac{n+1}{n-r}$$

for $2 \leq r \leq n-1$, it can be easily shown that the other root, viz.,

$$\frac{1}{2(n+1)}[\{(n-r)^2 + 4(r-1)(n+1)\}^{\frac{1}{2}} - (n-r)],$$

(say, $= H$), lies between 0 and 1. So by considering the root H in (5.2) and solving for y, we get the mode of $Y_{r:n}\ (2 \leq r \leq n-1)$ as

$$m_{r:n} = \mu \ln\left\{\frac{1+H}{1-H}\right\}.$$

Noting that $H = 0$ for $r = 1$ and $H = h$ for $r = n$, we can combine all the cases and say that the distribution of $Y_{r:n}$ is uni-modal with mode of $Y_{r:n}\ (1 \leq r \leq n)$ given by

$$m_{r:n} = \mu \ln\left\{\frac{1+H}{1-H}\right\},$$

where

$$H=\frac{1}{2(n+1)}[\{(n-r)^2+4(r-1)(n+1)\}^{\frac{1}{2}}-(n-r)].$$

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