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Inference on $P(Y<X)$ in an Exponential Distribution

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Abstract

Inference for probability $P(Y<X)$ in two parameter exponential distribution will be considered when the scale parameters are known or not : point and interval estimations, and test for a null hypothesis.

Keywords : exponential distribution, point and interval estimation, test for hypothesis

1. Introduction

A two-parameter exponential distribution is given by

$$f(x;\mu,\sigma)=\frac{1}{\sigma}e^{-(x-\mu)/\sigma},\quad x>\mu,\quad \text{where}\quad \sigma>0,\ \mu\in R, \tag{1.1}$$

it will be denoted $X\sim \text{EXP}(\mu,\sigma)$.

It is very important for us to consider an exponential distribution in parametric inferences. Here we shall consider inference for $P(Y<X)$ in two parameter exponential distribution.

The probability that a Weibull random variable Y is less than another independent Weibull random variable X was considered(McCool(1991)). Many other authors have considered the probability $P(Y<X)$, where X and Y are independent random variables.

The problem of estimating and of drawing inferences about, the probability that

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a random variable Y is less than an independent random variable X, arises in a reliability.

When Y represents the random value of a stress that a device will be subjected to in service and X represents the strength that varies from item to item in the population of devices, then the reliability R, i.e. the probability that a randomly selected device functions successfully, is equal to $P(Y<X)$. The same problem also arises in the context of statistical tolerance where represents, say, Y the diameter of a draft and X the diameter of a bearing that is to be mounted on the shaft. The probability that the bearing fits without interference is then $P(Y<X)$.

In biometrics, Y represents a patient's remaining years of life if treated with drug A and X represents the patient's remaining years when treated with drug B. If the choice of drug is left to the patient, person's deliberations will center on whether $P(Y<X)$ is less than or greater than 1/2.

Here, we shall consider inferences on $P(Y<X)$ in two parameter exponential distribution when the scale parameters are known or not : point and interval estimations, and test for a null hypothesis.

2. Inference on P(X<Y)

Let X and Y be independent two parameter exponential random variables, $X\sim$ EXP(μ_x, σ_x) and $Y\sim$EXP(μ_y, σ_y), respectively.

Then,
$$P(X<Y) = \int\int_{\mu_y<y<x} f_Y(y;\mu_y,\sigma_y)\cdot f_X(x;\mu_x,\sigma_x)dx$$
$$= 1-\frac{e^{\delta/\sigma_y}}{1+\sigma_x/\sigma_y}, \quad \text{where } \delta=\mu_y-\mu_x. \tag{2.1}$$

where $f_X(x)$ and $f_Y(y)$ are the density functions of X and Y , respectively.

To consider inferences on $P(X<Y)$, assume $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be two independent random samples from $X\sim \text{EXP}(\mu_x,\sigma_x)$ and $Y\sim \text{EXP}(\mu_y,\sigma_y)$, respectively.

Then the MLE $\hat{\delta}$ of δ is

$$\hat{\delta}=\widehat{\mu}_y-\widehat{\mu}_x=Y_{(1)}-X_{(1)}, \tag{2.2}$$

where $X_{(1)}$ and $Y_{(1)}$ are the first order statistics of $X_i's$ and $Y_i's$, respectively.

By the result of Johnson, etal.(1995),

Fact 1. (a) $X_{(1)}$ follows an exponential distribution with a location parameter μ_x and a scale parameter σ_x/m.

(b) If $X_1, X_2, \cdots, X_m$ are iid exponential distributions with a scale parameter σ and a location parameter μ, then $\sum_{i=1}^{m}(X_i - X_{(1)})$ follows a gamma distribution with a shape parameter $m-1$ and a scale parameter σ.

(c) If a random variable X follows a gamma distribution with a shape parameter α and a scale parameter β, then $E\left(\frac{1}{X^k}\right) = \frac{\Gamma(\alpha-k)}{\Gamma(\alpha)\beta^k}$, for $\alpha > k$.

From Fact 1(a), we can obtain the expectation and variance of $\hat{\delta}$:

$$E(\hat{\delta}) = \delta + \frac{\sigma_y}{n} - \frac{\sigma_x}{m} \quad \text{and} \quad Var(\hat{\delta}) = \frac{\sigma_x^2}{m^2} + \frac{\sigma_y^2}{n^2}. \tag{2.3}$$

Let $D \equiv Y_{(1)} - X_{(1)}$. Then we can obtain the density function of D :

$$f_D(d) = \begin{cases} \frac{mn}{n\sigma_x + m\sigma_y} e^{-\frac{n}{\sigma_y}(d-\delta)}, & \text{if } d \geq \delta \\ \frac{mn}{n\sigma_x + m\sigma_y} e^{-\frac{m}{\sigma_x}(\delta-d)}, & \text{if } d < \delta \end{cases}. \tag{2.4}$$

2-A. When the scale parameters $\sigma_x = \sigma_y = \sigma_0$ is known

From the result (2.1),

$$R = P(X < Y) = 1 - \frac{1}{2} e^{\delta/\sigma_0}, \qquad \delta = \mu_y - \mu_x.$$

Then the probability depends on δ only, Because R is a monotone function in δ, inference on δ is equivalent to inference on R. We hereafter confine attention to the parameter δ (see McCool(1991)).

When the scale parameters $\sigma_x = \sigma_y = \sigma_0$ is known, let $T = D - \delta$. Then from the pdf (2.4) of D, we have the pdf of T :

$$f_T(t)=\begin{cases}\dfrac{m}{m+n}\cdot\dfrac{n}{\sigma_0}e^{\frac{-n}{\sigma_0}t}, & \text{if } t\geq 0\\ \dfrac{n}{m+n}\cdot\dfrac{m}{\sigma_0}e^{\frac{m}{\sigma_0}t}, & \text{if } t<0.\end{cases} \tag{2.5}$$

Based on a pivotal quantity T, we shall consider an $(1-p_1-p_2)100\%$ confidence interval of δ. For a given $0<p_1<1$, there exists an b_1 such that $p_1=\int_{-\infty}^{b_1}\frac{n}{m+n}\cdot\frac{m}{\sigma_0}e^{\frac{m}{\sigma_o}t}dt$, and hence,

$$b_1=-\frac{\sigma_0}{2m}\cdot\chi^2_{2,\ \frac{m+n}{n}p_1}, \tag{2.6}$$

where $\alpha\equiv\int_{\chi^2_{2,\alpha}}^{\infty}\chi^2_2(t)dt$, $\chi^2_2(t)$ is the pdf of chi-square distribution of df 2.

For another given $0<p_2<1$, there exists an b_2 such that $p_2=\int_{b_2}^{\infty}\frac{m}{m+n}\cdot\frac{n}{\sigma_0}e^{-\frac{n}{\sigma_o}t}dt$, and

hence,
$$b_2=\frac{\sigma_0}{2n}\cdot\chi^2_{2,\ \frac{m+n}{m}p_2}. \tag{2.7}$$

Therefore, $(Y_{(1)}-X_{(1)}-b_2,\ Y_{(1)}-X_{(1)}-b_1)$ is an $(1-p_1-p_2)100\%$ confidence interval of δ in two parameter exponential distribution.

Next We wish to test the null hypothesis H_0 : $\mu_x=\mu_y$ against H_1 : $\mu_x\neq\mu_y$.

Let $\Theta=\{(\mu_x,\mu_y)\mid\mu_x\in R\ ,\ \mu_y\in R\}$ } and $\theta=(\mu_x,\mu_y)$.

Then the joint pdf of $(X_1,\cdots,X_m,Y_1,\cdots,Y_n)$ is

$$L(\theta)=f_\theta(x,y)=\prod_{i=1}^{m}\frac{1}{\sigma_0}e^{-\frac{1}{\sigma_0}(x_i-\mu_x)}\cdot\prod_{i=1}^{n}\frac{1}{\sigma_0}e^{-\frac{1}{\sigma_0}(y_i-\mu_y)},\ \text{for all }\ x_i>\mu_x,\ y_i>\mu_y.$$

From the likelihood function, we can obtain the MLE's of μ_x and μ_y,

$$\widehat{\mu}_x=X_{(1)}\quad\text{and}\quad\widehat{\mu}_y=Y_{(1)}.$$

If $\mu_x=\mu_y=\mu$, then the MLE of μ is

$$\widehat{\mu}=\min(X_{(1)},Y_{(1)})=(Y_{(1)}+X_{(1)}-\mid Y_{(1)}-X_{(1)}\mid)/2\ .$$

From definition of a likelihood ratio test(Rohatgi(1976)), the likelihood ratio test function can be obtained:

$$\lambda(x,y) = \exp(-\ |D|\ (\frac{m}{2\sigma_0} + \frac{n}{2\sigma_0}) + D(\frac{m}{2\sigma_0} - \frac{n}{2\sigma_0})\ ),$$
$$\text{where,} \quad D = Y_{(1)} - X_{(1)}.$$

Therefore, $\lambda(x,y) < c$ is equivalent to $D < b_1$ or $D > b_2$. (2.8)

Under $H_0 : \mu_x = \mu_y$, i.e. $\delta = 0$, we hold $T = D - \delta = D$, and hence, for given $0 < \alpha < 1$ we can find b_1 and b_2 of (2.8), through the results (2.6) and (2.7) if $p_1 = p_2 = \alpha/2$.

2-B. When the scale parameters $\sigma_x = \sigma_y = \sigma$ is unknown

First we wish to know whether two scale parameters are equal or not:

To test the null hypothesis $H_0 : \sigma_x = \sigma_y = \sigma$ against $H_1 : \sigma_x \neq \sigma_y$, $\mu_x \in R$, $\mu_y \in R$

Let $\Theta = \{(\sigma_x, \sigma_y, \mu_x, \mu_y) \mid \sigma_x > 0, \sigma_y > 0, \ \mu_x \in R, \ \mu_y \in R\}$ and $\theta = (\sigma_x, \sigma_y, \mu_x, \mu_y)$.

Then the joint pdf of $(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$ is

$$L(\theta) = f_\theta(x,y) = \prod_{i=1}^{m} \frac{1}{\sigma_x} e^{-\frac{1}{\sigma_x}(x_i - \mu_x)} \cdot \prod_{i=1}^{n} \frac{1}{\sigma_y} e^{-\frac{1}{\sigma_y}(y_i - \mu_y)}, \text{ for all } x_i > \mu_x, \ y_i > \mu_y.$$

Differentiating with respect to σ_x and σ_y, we can obtain the MLE's

$$\widehat{\sigma_x} = \frac{1}{m} \sum_{i=1}^{m} X_i, \quad \widehat{\sigma_y} = \frac{1}{n} \sum_{i=1}^{n} Y_i, \text{ and } \widehat{\mu_x} = X_{(1)} \text{ and } \widehat{\mu_y} = Y_{(1)}.$$

If $\sigma_x = \sigma_y = \sigma$, then the MLE of σ is

$$\hat{\sigma} = \frac{1}{n+m} (\sum_{i=1}^{m} (X_i - \widehat{\mu_x}) + \sum_{i=1}^{n} (Y_i - \widehat{\mu_y})). \quad (2.9)$$

From definition of a likelihood ratio test(Rohatgi(1976)), the likelihood ratio test function can be obtained :

$$\lambda(x,y) = (\frac{\widehat{\sigma_x}}{\hat{\sigma}})^m \cdot (\frac{\widehat{\sigma_y}}{\hat{\sigma}})^n = (\frac{m+n}{m})^m \cdot (\frac{m+n}{n})^n \cdot (\frac{1}{1+1/U})^m \cdot (\frac{1}{1+U})^n,$$

where $U \equiv \dfrac{\sum_{i=1}^{m}(X_i - X_{(1)})}{\sum_{i=1}^{n}(Y_i - Y_{(1)})}$.

Therefore, $\lambda(x, y) < c$ is equivalent to $U < u_1$ or $U > u_2$. (2.10)

From Fact 1(b) and the results of Rohatgi(1976), we have the followings;

Fact 2. (a) $Z \equiv \dfrac{2\sum_{i=1}^{m}(X_i - X_{(1)})}{\sigma_x}$ and $W \equiv \dfrac{2\sum_{i=1}^{m}(Y_i - Y_{(1)})}{\sigma_y}$ follows chi-square distribution with df's 2(m-1) and 2(n-1), respectively.

(b) The random variables Z and W are independent.

Under $H_0 : \sigma_x = \sigma_y = \sigma$, from Fact 2, $U \equiv \dfrac{\sum_{i=1}^{m}(X_i - X_{(1)})}{\sum_{i=1}^{n}(Y_i - Y_{(1)})}$ follows a F-distribution with df's 2(m-1) and 2(n-1). And hence, for a given $0 < \alpha < 1$, $u_2 = F_{\alpha/2}(2(m-1), 2(n-1))$ and $u_1 = 1/F_{\alpha/2}(2(n-1), 2(m-1))$, from (2.10).

If $\sigma_x = \sigma_y = \sigma$, then from the result (2.1),

$$R = P(X < Y) = 1 - \frac{1}{2} e^{\delta/\sigma}, \quad \text{where } \delta = \mu_y - \mu_x.$$

Let $\beta \equiv \delta/\sigma$. Then, an estimator of β is defined as :

$$\hat{\beta} \equiv \hat{\delta}/\hat{\sigma} = \frac{(m+n)(Y_{(1)} - X_{(1)})}{\sum_{i=1}^{m}(X_i - X_{(1)}) + \sum_{i=1}^{n}(Y_i - Y_{(1)})}, \quad \text{from results (2.2) and (2.9).}$$

From the results (2.3) and Fact 1(c), we can obtain the followings :

$$E(\hat{\beta}) = \beta + \frac{3}{m+n-3}\beta + \frac{m^2 - n^2}{mn(m+n-3)},$$

and

$$Var(\hat{\beta}) = \frac{(m+n)^2(m^2+n^2)}{m^2 n^2 (m+n-3)^2 (m+n-4)}.$$

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