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# Approximate MLEs for Exponential Distribution Under Multiple Type-II Censoring

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## Abstract

When the available sample is multiply Type-II censored, the maximum likelihood estimators of the location and the scale parameters of two-parameter exponential distribution do not admit explicitly. In this case, we propose some approximate maximum likelihood estimators by approximating the likelihood equations appropriately. We present an example to illustrate these estimation methods.

## 1. Introduction

The exponential distributions have been used as models in analyzing life-time data quite extensively; for example, see Lawless (1982), Kambo (1978), Balakrishnan and Cohen (1991), Kang et al. (1997), and Kang and Cho (1998) etc. Most works are based on the assumption that the sample is either Type-II right censored or Type-II doubly censored, and some are concerned with Type-I censored samples. Balakrishnan (1990), Balasubramanian and Balakrishnan (1992), Fei and Kong (1994), and Upadhyay et al. (1996) considered the inference for the exponential distribution under multiple Type-II censoring. They obtained several point estimation methods for the one-parameter as well as two-parameter exponential distribution.

By considering the likelihood equations based on multiply Type-II censored samples and noting that they do not admit explicit maximum likelihood estimators, we propose some explicit estimators derived by appropriately approximating the likelihood equations.

Multiply Type-II censored samples may arise in practice in different ways. They may arise in life-testing experiments when the failure times of some units

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were not observed due to mechanical or experimental difficulties. Another situation where multiply censored samples arise naturally is when some units failed between two points of observation with exact times of failure of these units unobserved.

In this paper, we derive several type point estimators of the location and the scale parameters in an explicit form for the general case when the available sample is multiply Type-II censored.

## 2. Approximate Maximum Likelihood Estimators

Consider a two-parameter exponential distribution with the probability density function (pdf)

$$f(x\,;\theta,\sigma)=\frac{1}{\sigma}e^{-(x-\theta)/\sigma},\qquad x\geq\theta,\ \sigma>0 \tag{2.1}$$

and the cumulative distribution function (cdf)

$$F(x\,;\theta,\sigma)=1-e^{-(x-\theta)/\sigma},\qquad x\geq\theta,\ \sigma>0 \tag{2.2}$$

where $\theta$ is the warranty time, $\sigma$ is the residual mean life, and $\mu=\theta+\sigma$ is the expected life-time.

Suppose $n$ items are placed on a life-testing experiment and that the $a_1$th, $a_2$th, ..., $a_s$th failure-times are only made available, where

$$1\leq a_1<a_2<\cdots<a_s\leq n.$$

That is,

$$X_{a_1:n}\leq X_{a_2:n}\leq\cdots\leq X_{a_s:n} \tag{2.3}$$

is the multiply Type-II censored sample available from (2.1)

Let $a_0=0$, $a_{s+1}=n+1$, $F(x_{a_0:n})=0$, $F(x_{a_{s+1}:n})=1$, then the likelihood function based on the multiply Type-II censored sample (2.3) is given by

$$\begin{aligned}L&=\frac{n!}{\prod_{j=1}^{s+1}(a_j-a_{j-1}-1)!}\left[\prod_{j=1}^{s+1}[F(Z_{a_j:n})-F(Z_{a_{j-1}:n})]^{a_j-a_{j-1}-1}\right]\frac{1}{\sigma^s}\prod_{j=1}^{s}f(Z_{a_j:n})\\&=\frac{1}{\sigma^s}\frac{n!}{\prod_{j=1}^{s+1}(a_j-a_{j-1}-1)!}[F(Z_{a_1:n})]^{a_1-1}\left[\prod_{j=2}^{s}[F(Z_{a_j:n})-F(Z_{a_{j-1}:n})]^{a_j-a_{j-1}-1}\right]\\&\quad\times[1-F(Z_{a_s:n})]^{n-a_s}\prod_{j=1}^{s}f(Z_{a_j:n})\end{aligned} \tag{2.4}$$

where $Z_{i:n}=(X_{i:n}-\theta)/\sigma$, and $f(z)=e^{-z}$ and $F(z)=1-e^{-z}$ are the pdf and the cdf of the standard exponential distribution respectively.

Balasubramanian and Balakrishnan (1992) derived the maximum likelihood estimator of $\theta$ as follows;

$$\hat{\theta}=\begin{cases} X_{1:n}, & \text{if } a_1=1 \\ X_{a_1:n}+\hat{\sigma}\ln\left(\frac{n-a_1+1}{n}\right), & \text{if } a_1>1 \end{cases}$$

Since this maximum likelihood estimator is the function of the MLE of the scale parameter $\sigma$, we now consider an unbiased estimator of $\theta$ that is function of order statistics as follows;

$$\widehat{\theta_1}=\frac{1}{h(a_2)-h(a_1)}\left[h(a_2)X_{a_1:n}-h(a_1)X_{a_2:n}\right]$$

where

$$h(a)=\sum_{j=1}^{a}(n-j+1)^{-1}$$

Also we can derive the other estimator by minimizing the mean squared error (MSE) among the class of estimators of the form $cX_{a_1:n}+(1-c)X_{a_2:n}$ where $c$ is constant.

$$\widehat{\theta_2}=cX_{a_1:n}+(1-c)X_{a_2:n}$$

where

$$c=\frac{g(a_1)-g(a_2)-h^2(a_2)+h(a_1)h(a_2)}{g(a_1)-g(a_2)-[h(a_1)-h(a_2)]^2}$$

$$g(a)=\sum_{j=1}^{a}(n-j+1)^{-2}$$

From (2.4), the likelihood equation for $\sigma$ is obtained as

$$\begin{aligned}\frac{\partial \ln L}{\partial\sigma} &= -\frac{1}{\sigma}\Bigg[s+(a_1-1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}Z_{a_1:n}-(n-a_s)Z_{a_s:n}-\sum_{j=1}^{s}Z_{a_j:n} \\ &\quad+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(Z_{a_j:n})Z_{a_j:n}-f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\Bigg] \\ &=0 \end{aligned} \qquad (2.5)$$

Equation (2.5) does not admit an explicit solution for $\sigma$. But we can expand the functions $\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}$ and $\frac{f(Z_{a_j:n})Z_{a_j:n}-f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}$ as follows;

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}\approx\alpha-\beta Z_{a_1:n} \qquad (2.6)$$

$$\frac{Z_{a_j:n} f(Z_{a_j:n}) - Z_{a_{j-1}:n} f(Z_{a_{j-1}:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \approx \alpha_j + \beta_j Z_{a_j:n} + \gamma_j Z_{a_{j-1}:n} \tag{2.7}$$

where

$$p_i = \frac{i}{n+1}, \quad q_i = 1 - p_i$$

$$\xi_r = F^{-1}(p_\gamma) = -\ln q_\gamma$$

$$\alpha = \frac{f(\xi_{a_1})}{p_{a_1}} \left[ 1 + \xi_{a_1} + \frac{f(\xi_{a_1})}{p_{a_1}} \xi_{a_1} \right]$$

$$\beta = \frac{f(\xi_{a_1})}{{p_{a_1}}^2} [p_{a_1} + f(\xi_{a_1})]$$

$$\alpha_j = \frac{{\xi_{a_j}}^2 f(\xi_{a_j}) - {\xi_{a_{j-1}}}^2 f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} + \left( \frac{\xi_{a_j} f(\xi_{a_j}) - \xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right)^2$$

$$\beta_j = \frac{f(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}} \left[ 1 - \xi_{a_j} - \frac{\xi_{a_j} f(\xi_{a_j}) - \xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right]$$

$$\gamma_j = \frac{-f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \left[ 1 - \xi_{a_{j-1}} - \frac{\xi_{a_j} f(\xi_{a_j}) - \xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right]$$

Upon using the approximation of (2.6) and (2.7) in the likelihood equation of (2.5), we derive two approximate maximum likelihood estimators of $\sigma$ to be

$$\widehat{\sigma}_i = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \quad i = 1, 2 \tag{2.8}$$

where

$$A = s + \sum_{j=2}^{s} (a_j - a_{j-1} - 1) \alpha_j$$

$$\begin{aligned} B = {} & (a_1 - 1) \alpha X_{a_1:n} - (n - a_s) X_{a_s:n} - \sum_{j=1}^{s} X_{a_j:n} \\ & + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\beta_j X_{a_j:n} + \gamma_j X_{a_{j-1}:n}) \\ & - [(a_1 - 1)\alpha - (n - a_s) - s + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\beta_j + \gamma_j)] \widehat{\theta}_i \end{aligned}$$

$$C = (a_1 - 1) \beta (X_{a_1:n} - \widehat{\theta}_i)$$

It is well known that the expectation and the variance of the $i$th order statistic, and the covariance of the $i$th and the $j$th order statistics from the two-parameter exponential distribution with pdf (2.1) are given by

$$E(X_{i:n}) = \mu + \sigma \sum_{j=1}^{i} (n - j + 1)^{-1} \tag{2.9}$$

$$\mathrm{Var}(X_{i:n}) = \sigma^2 \sum_{k=1}^{i} (n - k + 1)^{-2}$$

$$= \mathrm{Cov}(X_{i:n},\ X_{j:n}),\quad i \le j \qquad (2.10)$$

Now from (2.9), we can obtain the means and variances of the estimators of the location parameter $\theta$ as follows

$$E(\widehat{\theta}_1) = \theta$$

$$E(\widehat{\theta}_2) = \theta + [ch(a_1) + (1-c)h(a_2)]\sigma$$

$$\mathrm{Var}(\widehat{\theta}_1) = \mathrm{MSE}(\widehat{\theta}_1)$$

$$= \frac{1}{[h(a_2) - h(a_1)]^2}[\{h(a_2) - 2h(a_1)\}h(a_2)g(a_1) + h^2(a_1)g(a_2)]\sigma^2$$

$$\mathrm{Var}(\widehat{\theta}_2) = [c^2 g(a_1) + (1-c)^2 g(a_2) + 2c(1-c)g(a_1)]\sigma^2$$

## 3. Illustrative Example

We consider the data presented in Balasubramanian and Balakrishnan (1992). Thirty items were placed on a life-testing experiment and their times-to-fail (in hours) were recorded as given below:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.961 | 0.990 | 1.565 | 2.031 | 2.204 | 2.340 | 3.642 | 6.008 | 6.538 | 7.145 |
| - | - | - | 11.937 | 15.433 | 18.234 | 18.307 | 22.096 | - | - |
| - | 28.799 | 30.692 | 30.737 | 33.702 | 34.245 | - | - | - | - |

Here, some middle observations were not recorded exactly due to experimental difficulties and the last four observations were censored since the experiment was stopped as soon as the twenty sixth item failed. This was a simulated data set from a one-parameter exponential distribution with $\sigma = 20$

Some point estimates have been provided by Balasubramanian and Balakrishnan (1992). In fact, the best linear unbiased estimates of $\theta$, $\sigma$, and $\mu$ are

$$\theta^* = 0.3081,\quad \sigma^* = 19.5863,\quad \mu^* = 19.8944$$

and the AMLE method yields

$$\hat{\theta} = 0.9610,\quad \hat{\sigma} = 18.8623,\quad \hat{\mu} = 19.8233$$

respectively. In this case, our proposed estimates are

$$\widehat{\theta}_1 = 0.9330,\quad \widehat{\sigma}_1 = 18.8946,\quad \widehat{\mu}_1 = 19.8276$$

$$\widehat{\theta}_2 = 0.9470,\quad \widehat{\sigma}_2 = 18.8784,\quad \widehat{\mu}_2 = 19.8254$$

Since the AMLE $\hat{\theta}$ overestimate the location parameter $\theta$, the proposed estimates can be more reseanable than the AMLE provided by Balasubramanian and Balakrishnan (1992).

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