Journal of Korean
Data & Information Science Society
2003, Vol. 14, No.3 pp. 661~671

Intrinsic Priors for Testing Two Lognormal Populations with the Fractional Bayes Factor

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Abstract

The Bayes factors with improper noninformative priors are defined only up to arbitrary constants. So, it is known that Bayes factors are not well defined due to this arbitrariness in Bayesian hypothesis testing and model selections. The intrinsic Bayes factor by Berger and Pericchi (1996) and the fractional Bayes factor by O'Hagan (1995) have been used to overcome this problems. This paper suggests intrinsic priors for testing the equality of two lognormal means, whose Bayes factors are asymptotically equivalent to the corresponding fractional Bayes factors. Using proposed intrinsic priors, we demonstrate our results with real example and a simulated dataset.

KeyWords : Bayes factor, Fractional Bayes factor, Intrinsic prior, Noninformative prior, Reference prior

1. Introduction

It has been well known that Bayes factors with proper priors have been very successful in testing or model selection problems. However, in Bayesian analysis, limited informations and time restrictions often force to the use of noninformative priors such as Jeffrey's priors or reference priors. These noninformative priors are usually improper density and the Bayes factors under improper priors are not well defined because these priors are defined only up to arbitrary constants.

Suppose that $\boldsymbol{x}=(x_1, x_2, \cdots, x_n)$ are random samples from population with a probability density $f(\boldsymbol{x}|\theta_i)$, where θ_i is a vector of unknown parameters,

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$i=1,2$. Let Θ_i be the parameter space for θ_i and $\pi_i^N(\theta_i)$ be the improper prior density, $i=1,2$. The Bayes factor B_{21}^N of model M_2 to model M_1 is defined by

$$B_{21}^N = \frac{m_2^N(x)}{m_1^N(x)} = \frac{\int_{\Theta_2} f(x \mid \theta_2)\pi_2^N(\theta_2)d\theta_2}{\int_{\Theta_1} f(x \mid \theta_1)\pi_1^N(\theta_1)d\theta_1}, \tag{1.1}$$

where $m_1^N(x)$ and $m_2^N(x)$ are the marginal densities under model M_1 and model M_2, respectively. Since $\pi_1^N(\theta_1)$ and $\pi_2^N(\theta_2)$ are improper, the Bayes factor in (1.1) contains arbitrary constants, say c_1 and c_2. So, the resulting Bayes factor is not well defined.

Several authors including Geisser and Eddy (1979), Spiegalhalter and Smith (1982), and San Martini and Spezzaferri (1984) have made efforts to overcome this problems. O'Hagan (1995) introduced the fractional Bayes factor (FBF) and Berger and Pericchi (1996) suggested the intrinsic Bayes factor (IBF) as model selection criterion. Two methods have been often served as default Bayes factors.

The IBF is the method for removing the arbitrariness by a subset of data called a training sample. There are several papers using IBF (cf Kim (2000); Kim and Sun (2000); Kim, Kang and Kim (2000)). But, IBF approach has some difficulties due to its considerable computational expense for large sample sizes and unstability in small sample. Another useful criterion, FBF, is the method for removing the arbitrariness by a portion of the likelihood, which is computed by exponentiating the likelihood to a power b, where $0 \leq b \leq 1$. It is well defined as in the IBF method. Moreover, it does not require a heavy computation and thus much more effective in the sense of computation. Kim and Kim (2000) proposed a Bayesian testing for the comparison of two exponential means using proper intrinsic priors, whose Bayes factors are asymptotically equivalent to the corresponding FBF's. Bae, Kim and Kim (2000) derived proper intrinsic priors to test the equality of two independent normal means with unknown variance.

In this paper, we consider the testing problem for comparing two lognormal means. Particularly, we derive proper intrinsic priors, whose Bayes factors are asymptotically equivalent to the corresponding FBF's. The outline of this paper is as follows. In section 2, we review the concepts of the fractional Bayes factor and the intrinsic prior. In section 3, we derive the intrinsic priors and calculate the Bayes factors for the purpose of comparing two independent lognormal means. Some numerical results with real dataset and simulation results are given in section 4.

2. Preliminaries

It has known that the Bayes factor B_{21}^N in (1.1) involves arbitrary constants. Recently two methods for removing this arbitrariness have been proposed, which are to use a subset of data, a training sample, and a portion of the likelihood, a fraction b.

The intrinsic Bayes factor introduced by Berger and Pericchi (1996) is to use the part of the data as the minimal training sample. A training sample, $x(l)$ is called a minimal training sample if it has the minimal sample size to guarantee $0 < m_i^N(x(l)) < \infty$ for all model M_i, $i=1,2$. O'Hagan (1995) proposed the fractional Bayes factor for removing the arbitrariness in (1.1) by using a fractional part of the entire likelihood with the fraction b instead of training sample, and suggested the choice of the fraction.

Definition 1. The fractional Bayes factor (FBF) of model M_2 to model M_1 is

$$B_{21}^F = B_{21}^N \times CF_{12}(b), \tag{2.1}$$

where the correction factor $CF_{12}(b)$ is defined as

$$CF_{12}(b) = \frac{\int_{\Theta_1} L_1^b(\theta_1)\pi_1^N(\theta_1)\,d\theta_1}{\int_{\Theta_2} L_2^b(\theta_2)\pi_2^N(\theta_2)\,d\theta_2}.$$

Here, $L_i(\theta_i)$ is the likelihood function under model M_i, $i=1,2$ and b is the fraction of the likelihood, $0 \le b \le 1$. A commonly suggested choice is $b=m/n$, where m is the size of the minimal training sample and n is the sample size. We will use this choice in our problem.

It is of interest to find proper priors, often called intrinsic priors, so that the Bayes factors using proper priors will be asymptotically equivalent to the corresponding FBF's. This issue was suggested by Berger and Pericchi (1996). To derive the intrinsic priors, we need the following conditions.

Condition. As the sample size grows to infinity, the following folds:

1. Under model M_1, $\widehat{\theta}_1 \to \theta_1$, $\widehat{\theta}_2 \to \phi_2(\theta_1)$,
2. Under model M_2, $\widehat{\theta}_2 \to \theta_2$, $\widehat{\theta}_1 \to \phi_1(\theta_2)$,

where $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are the maximum likelihood estimators (MLE's) under model M_1 and model M_2. Then, a set of intrinsic priors denoted by $(\pi_1^I(\theta_1), \pi_2^I(\theta_2))$ is a solution of the following equations:

$$\begin{cases} \dfrac{\pi_2^I(\phi_2(\theta_1))\,\pi_1^N(\theta_1)}{\pi_2^N(\phi_2(\theta_1))\,\pi_1^I(\theta_1)} = B_1^*(\theta_1), \\ \dfrac{\pi_2^I(\theta_2)\,\pi_1^N(\phi_1(\theta_2))}{\pi_2^N(\theta_2)\,\pi_1^I(\phi_1(\theta_2))} = B_2^*(\theta_2), \end{cases} \tag{2.2}$$

where for $i=1,2$, $B_i^*(\theta_i) = \lim_{N\to\infty} CF_{12}(b)$ under model M_i.

The noninformative priors $\pi_1^N(\theta_1)$ and $\pi_2^N(\theta_2)$ are called starting priors. The solutions of the above equations are not necessarily unique, nor proper. We will find proper priors for given starting priors. Once we derive proper intrinsic priors, the fractional Bayes factor B_{21}^F can be replaced by the ordinary Bayes factor B_{21}^I computed based on intrinsic priors.

3. Testing two lognormal means

The lognormal distribution has become increasingly widespread in life testing, reliability and other related fields of application. An random variable X such as failure or down time, is said to have a lognormal distribution if $Y=\log X$ is normally distributed. By means of a simple logarithmic transformation of variable, it can easily be shown that the lognoraml probability density function is given by

$$f(x \mid \mu, \eta) = \frac{1}{\sqrt{2\pi\eta}\,x}\exp\left\{-\frac{1}{2\eta}(\log x-\mu)^2\right\},\ 0<x<\infty.\ -\infty<\mu<\infty,\ 0<\eta<\infty,$$

where $\mu=E(\log X)$ and $\eta=Var(\log X)$. The lognormal distribution will be denoted as the $LN(\mu,\eta)$.

Suppose that we have independent random samples $X_{ij}\sim LN(\mu_i,\eta)$, $i=1,2$, $j=1,2,\cdots,n_i$, where η is common and unknown. We are interested in testing the equality of two lognormal means, that is,

$$M_1:\ \exp(\mu_1+\frac{\eta}{2}) = \exp(\mu_2+\frac{\eta}{2}) \quad v.s. \quad M_2:\ \exp(\mu_1+\frac{\eta}{2}) \neq \exp(\mu_2+\frac{\eta}{2}).$$

But, because the scale parameters η of two populations are common, it equals to test the equality of two parameters μ_1 and μ_2, that is,

$$M_1:\ \mu_1=\mu_2(=\mu) \quad v.s. \quad M_2:\ \mu_1\neq\mu_2.$$

Let for $\theta_1=(\mu,\eta)$ and $\theta_2=(\mu_1,\mu_2,\eta)$, $\Theta_1=\{(\mu,\eta)\mid -\infty<\mu<\infty, 0<\eta<\infty\}$ and $\Theta_2=\{(\mu_1,\mu_2,\eta)\mid -\infty<\mu_1,\mu_2<\infty, 0<\eta<\infty\}$. Let $Y_{ij}=\log X_{ij}$ and $T_i=\prod_{j=1}^{n_i} X_{ij}$ for $j=1,2,\cdots,n_i$, $i=1,2$. Let $N=n_1+n_2$ with $n_1/N\to p$ as $N\to\infty$.

From now on, we will find a set of intrinsic priors $(\pi_1^I(\theta_1), \pi_2^I(\theta_2))$, which are proper over $\theta_i \in \Theta_i$, $i=1,2$. We start with the reference priors for model M_1 and model M_2 given by as follows:

$$\pi_1^N(\mu, \eta) = \pi_2^N(\mu_1, \mu_2, \eta) = \frac{1}{\eta}, \quad 0 < \eta < \infty .$$

Then the Bayes factor under the reference priors π_1^N and π_2^N is given by

$$B_{21}^N = \frac{\sqrt{\pi}\sqrt{N}}{\sqrt{n_1 n_2}} \frac{\Gamma((N-2)/2)}{\Gamma((N-1)/2)} \frac{S^{N-1}}{S_{12}^{N-2}}, \tag{3.1}$$

where $S^2 = \sum_{i=1}^{2}\sum_{j=1}^{n_i}(Y_{ij} - \overline{Y})^2$, $\overline{Y} = \frac{1}{N}\sum_{i=1}^{2}\sum_{j=1}^{n_i} Y_{ij}$, $S_{12}^2 = \sum_{i=1}^{2}\sum_{j=1}^{n_i}(Y_{ij} - \overline{Y}_i)^2$, $\overline{Y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij}$, $i=1,2$.

Remark. Note that if we take one observation from each population as a training sample, the marginal density $m_1^N(x(l))$ is finite, but $m_2^N(x(l))$ is not finite. So in our situation, the size of the minimal training sample is m=3.

The correction factor at $b=3/N$ in (2.1) is given by

$$CF_{12}\left(\frac{3}{N}\right) = \frac{1}{\pi} \frac{\sqrt{n_1 n_2}}{N} \frac{\sqrt{S_{12}^2/N}}{S^2/N}, \tag{3.2}$$

where S^2 and S_{12}^2 are in (3.1).

To find proper priors, we first find the asymptotic behavior of MLE's of $\widehat{\theta}_1 = (\overline{Y}, S^2/N)$ and $\widehat{\theta}_2 = (\overline{Y}_1, \overline{Y}_2, S_{12}^2/N)$ under model M_1 and M_2.

Under model M_1: $\mu_1 = \mu_2 = \mu$, one can easily see that

$$E_{\theta_1}^{M_1}(\overline{Y}) = \mu, \quad E_{\theta_1}^{M_1}\left(\frac{S^2}{N}\right) = \left(1 - \frac{1}{N}\right)\eta$$

and $E_{\theta_1}^{M_1}(\overline{Y}_1) = E_{\theta_1}^{M_1}(\overline{Y}_2) = \mu$, $E_{\theta_1}^{M_1}\left(\frac{S_{12}^2}{N}\right) = \left(1 - \frac{2}{N}\right)\eta$.

Under model M_2: $\mu_1 \neq \mu_2$,

$$E_{\theta_2}^{M_2}(\overline{Y}) = \frac{n_1}{N}\mu_1 + \frac{n_2}{N}\mu_2, \quad E_{\theta_2}^{M_2}\left(\frac{S^2}{N}\right) = \left(1 - \frac{2}{N}\right)\eta + \frac{\eta}{N} + \frac{n_1 n_2}{N^2}(\mu_1 - \mu_2)^2$$

and $E_{\theta_2}^{M_2}(\overline{Y}_1) = \mu_1$, $E_{\theta_2}^{M_2}(\overline{Y}_2) = \mu_2$, $E_{\theta_2}^{M_2}\left(\frac{S_{12}^2}{N}\right) = \left(1 - \frac{2}{N}\right)\eta$.

Proposition 1. As $N \to \infty$, under model M_1: $\mu_1 = \mu_2 = \mu$,

$$\widehat{\theta}_1 = (\overline{Y}, S^2/N) \to \theta_1 = (\mu, \eta)$$

and

$$\widehat{\theta_2}=(\overline{Y_1},\ \overline{Y_2},\ S_{12}^2/N)\rightarrow(\mu,\mu,\eta)\equiv\phi_2(\theta_1).$$

Under model M_2: $\mu_1 \neq \mu_2$,

$$\widehat{\theta_1}=(\overline{Y},\ S^2/N)\rightarrow(p\mu_1+q\mu_2,\ \eta+pq(\mu_1-\mu_2)^2)\equiv\phi_1(\theta_2).$$

where $q=1-p$, and

$$\widehat{\theta_2}=(\overline{Y_1},\ \overline{Y_2},\ S_{12}^2/N)\rightarrow\theta_2=(\mu_1,\mu_2,\eta).$$

Now, we compute $B_1^*(\theta_1)$ and $B_2^*(\theta_2)$ in the intrinsic equations (2.2).

Proposition 2. The quantities $B_1^*(\theta_1)$ and $B_2^*(\theta_2)$ are

$$B_1^*(\theta_1)=\frac{\sqrt{pq}}{\pi}\cdot\frac{1}{\sqrt{\eta}}\ \text{ and }\ B_2^*(\theta_2)=\frac{\sqrt{pq}}{\pi}\cdot\frac{\sqrt{\eta}}{\eta+pq(\mu_1-\mu_2)^2}. \tag{3.3}$$

Proof. The results immediately follow from the strong law of large numbers.

Proposition 3. As $(\mu_1,\mu_2,\eta)\rightarrow(\mu,\mu,\eta)$,

$$B_2^*(\theta_2)\rightarrow B_1^*(\theta_1).$$

Proof. It is straightforward.

Now, from the results of Proposition 1 and 2 and the intrinsic equations given in (2.2), a set of the intrinsic priors is given by

$$\begin{cases}\pi_1^I(\mu,\eta)=g(\mu,\eta),\\ \pi_2^I(\mu_1,\mu_2,\eta)=\dfrac{\sqrt{pq}}{\pi\sqrt{\eta}}\times g(p\mu_1+q\mu_2,\eta+pq(\mu_1-\mu_2)^2),\end{cases} \tag{3.4}$$

where $g(\mu,\eta)$ is any proper density for $(\mu,\eta)\in\Theta_1$.

Theorem 1. The intrinsic prior $\pi_2^I(\mu_1,\mu_2,\eta)$ in (3.4) is proper on Θ_2.

Proof. Let $s=p\mu_1+q\mu_2$ and $t=\mu_1-\mu_2$. Then

$$\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\pi_2^I(\mu_1,\mu_2,\eta)\,d\mu_1 d\mu_2 d\eta$$
$$=\frac{\sqrt{pq}}{\pi}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{1}{\sqrt{\eta}}\cdot g(s,\eta+pqt^2)\,ds\,dt\,d\eta.$$

Let $k_1=\eta$ and $k_2=\sqrt{pq}\,t$. Then the above equation becomes

$$\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{1}{\sqrt{k_1}}\cdot g(s,k_1+k_2^2)\,ds\,dk_2\,dk_1.$$

And letting $u=\frac{k_2}{\sqrt{k_1}}$ and $\nu=\sqrt{k_1}$, then

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\nu\cdot g(s,\nu^2(1+u^2))\,ds\,d\nu\,du.$$

Finally, letting $w=\nu^2(1+u^2)$, then

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}g(s,w)\frac{1}{1+u^2}\,ds\,dw\,du$$

and since g is a probability density function on Θ_1, therefore

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+u^2}\,du=1.$$

This completes the proof.

Corollary 1. When $g(\mu,\eta)$ is given by $g(\mu,\eta)=g_1(\mu\mid\eta)g_2(\eta)$ with $\mu\mid\eta\sim N(\omega,\varkappa\eta)$ and $\eta\sim IG(\alpha,\beta)$, the set of intrinsic priors is given by

$$\begin{cases}\pi_1^I(\mu,\eta)=\dfrac{\beta^\alpha}{\sqrt{2\pi\varkappa}\Gamma(\alpha)}\left(\dfrac{1}{\eta}\right)^{\alpha+\frac{3}{2}}\exp\left[-\dfrac{1}{\eta}\left(\dfrac{(\mu-\omega)^2}{2\varkappa}+\beta\right)\right],\\ \pi_2^I(\mu_1,\mu_2,\eta)=\dfrac{\sqrt{pq}}{\pi\sqrt{\eta}}\times g(p\mu_1+q\mu_2,\eta+pq(\mu_1-\mu_2)^2),\end{cases}\tag{3.5}$$

where

$$g(p\mu_1+q\mu_2,\eta+pq(\mu_1-\mu_2)^2)=\frac{\sqrt{pq}}{\pi\sqrt{\eta}}\frac{\beta^\alpha}{\sqrt{2\pi\varkappa}\Gamma(\alpha)}\left(\frac{1}{\eta+pq(\mu_1-\mu_2)^2}\right)^{\alpha+3/2}$$
$$\times\exp\left[-\frac{1}{\eta+pq(\mu_1-\mu_2)^2}\left(\frac{(p\mu_1+q\mu_2-\omega)^2}{2\varkappa}+\beta\right)\right].$$

Corollary 2. Using the intrinsic priors in Corollary 1, the Bayes factor of model M_2 to model M_1 is given by

$$B_{21}^I=\frac{m_2^I(x)}{m_1^I(x)},\tag{3.6}$$

where

$$m_1^I(x)=\left(\frac{1}{2\pi}\right)^{N/2}\frac{1}{\sqrt{1+\varkappa N}}\frac{\beta^\alpha}{\Gamma(\alpha)}\frac{1}{T_1T_2}\frac{\Gamma(\alpha+N/2)}{\left(\frac{N(\omega-\bar{y})^2}{2(1+\varkappa N)}+\beta+\frac{s^2}{2}\right)^{\alpha+N/2}},$$

and

$$m_2^I(x) = \left(\frac{1}{2\pi}\right)^{N/2} \frac{2}{\pi} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{T_1 T_2}$$
$$\times \int_0^\infty \int_{-\infty}^\infty \left(\frac{1}{\nu}\right)^{N+2\alpha+1} \left(\frac{1}{1+u^2}\right)^{\alpha+1} \frac{1}{\sqrt{1+\varkappa N(1+u^2)}}$$
$$\times \exp\left(-\frac{s_{12}^2}{2\nu^2} - \frac{\beta}{\nu^2(1+u^2)} - R(u,\nu)\right) du d\nu,$$

respectively with

$$R(u,\nu) = \frac{1}{2\nu^2(1+\varkappa N(1+u^2))}\Big[n_1(\bar{y}_1 - \sqrt{\frac{q}{p}}u\nu - \omega)^2 + n_2(\bar{y}_2 + \sqrt{\frac{p}{q}}u\nu - \omega)^2$$
$$+ n_1 n_2 \varkappa(1+u^2)(\bar{y}_1 - \bar{y}_2 - \frac{u\nu}{\sqrt{pq}})^2\Big].$$

Proof. It is straightforward.

4. Numerical results

In this section, we will give some examples to compare the fractional Bayes factors and the Bayes factors using the intrinsic priors via the real dataset and the simulation.

Example 1. The data given here arose in test on the endurance of deep groove ball bearings (Lawless (1982)). These data were assumed to come from Weibull distribution. But a probability plot of the data showed them to also be consonant with a lognormal model. The data are the number of million revolutions before failure for each the 23 ball bearings in life test. Suppose that the data are divided into the following two groups to test the hypothesis model $M_1: \mu_1 = \mu_2$ and $M_2: \mu_1 \neq \mu_2$.

group 1	33.00, 45.60, 51.84, 51.96, 55.56, 67.80, 68.64, 68.64, 93.12, 105.84, 128.04
group 2	17.88, 28.92, 41.52, 42.12, 48.40, 54.12, 68.88, 84.12, 98.64, 105.12, 127.92, 173.40

By the logarithmic transformations of the given data, the data follow the normal distribution and T-test is used for comparing the equality of two means.

First, we obtain F-statistic to examine that the variances of two groups are equal. The F-statistic and the corresponding p-value are 0.3618 and 0.1024, and we can see that the variances of two groups are equal as we expected.

The value of well known T-statistic with pooled variance estimator is 0.2303. The corresponding p-value is 0.4101, and the model M_1 is accepted.

Now, we compute the fractional Bayes factor and the Bayes factor using the intrinsic priors in (3.6) with (α, β)=(0.01,0.01),(0.1,0.1),(1.0,1.0) and χ=1,5,10,50,100, assuming that $\omega=0$. The results are reported in Table 1.

Table 1. Bayes factors for testing $M_1: \mu_1=\mu_2$ $v.s.$ $M_2: \mu_1\neq\mu_2$

B^I_{21} \ B^F_{21}	χ	(α, β)		
		(0.01,0.01)	(0.1,0.1)	(1.0,1.0)
.18070	1	.23824	.23819	.23763
	5	.19424	.19538	.20220
	10	.17069	.17237	.18990
	50	.15944	.15929	.16870
	100	.15964	.15924	.16612

From Table 1, we can see that the Bayes factors with the intrinsic priors are nearly free to hyperparameters (α, β), but depend on χ slightly. For fairly large value of χ, the Bayes factors give almost same values of the fractional Bayes factors. Since the Bayes factors for all cases are less than 1, we may conclude that the difference between two groups is fairly small.

Example 2. We performed the simulation study for testing $M_1: \mu_1=\mu_2=0$ and $M_2: \mu_1\neq\mu_2$ to examine how well Bayes factors using the intrinsic priors approximate to the fractional Bayes factors. The random numbers are generated from lognormal distribution with $\mu_i=0$, $i=1,2$ and common parameter η=1. The simulation is replicated 100 times for some choices of (α, β) and χ, assuming that ω=0. We compute the relative errors $|B^F_{21}-B^I_{21}|/B^F_{21}$ and standard deviations of relative errors for (α, β)=(0.01,0.01),(0.1,0.1),(1.0,1.0) and χ=5,10,50. The values of χ are chosen by the intuitive reason that in Example 1, Bayes factors using the intrinsic priors for these values of χ gives almost same values of the fractional Bayes factors. The results are reported in Table 2. The numbers in parentheses are the standard deviations of the relative errors.

Table 2. Absolute relative errors between B_{21}^F and B_{21}^I

(n_1, n_2)	χ \ (α, β)	(0.01,0.01)	(0.1,0.1)	(1.0,1.0)
(10,10)	5	.10835(.01847)	.10791(.01841)	.10406(.02230)
(10,20)		.07541(.01751)	.07526(.01767)	.07387(.02010)
(20,20)		.06520(.02796)	.06534(.02799)	.06715(.02869)
(20,30)		.06091(.04268)	.06117(.04297)	.06481(.04486)
(30,30)		.08410(.05642)	.08453(.05677)	.08913(.06012)
(10,10)	10	.10861(.01840)	.10817(.01833)	.10431(.02223)
(10,20)		.07555(.01747)	.07539(.01763)	.07400(.02006)
(20,20)		.06527(.02796)	.06541(.02799)	.06721(.02871)
(20,30)		.06094(.04270)	.06120(.04298)	.06482(.04488)
(30,30)		.08411(.05643)	.08455(.05678)	.08913(.06016)
(10,10)	50	.10882(.01834)	.10838(.01827)	.10451(.02217)
(10,20)		.07566(.01744)	.07550(.01760)	.07410(.02004)
(20,20)		.06532(.02796)	.06547(.02799)	.06726(.02873)
(20,30)		.06097(.04271)	.06123(.04300)	.06484(.04490)
(30,30)		.08412(.05644)	.08456(.05679)	.08913(.06018)

We can see that relative errors are quite small for each simulated dataset. It is well known that the Bayes factors using noninformative priors are defined only up to arbitrary constants. The intrinsic Bayes factor and the fractional Bayes factor are proposed for removing this arbitrariness. The IBF approach have some advantages to be quite satisfactory in nonnested, as well as nested, model comparison and any distribution. However, the IBF has difficulty to require a heavy computation because the number of training sample might be enormous for large sample size. But FBF's are easier to compute than IBF's and more effective in computational time-consuming. The FBF approach has disadvantage to be inadequate for very small sample size. So, the FBF is limited than IBF in application range. However, FBF is also well defined and seems to be reasonably close to actual Bayes factors. The numerical results show that the FBF′s are asymptotically equivalent to the corresponding Bayes factors using the intrinsic priors.

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[received date : May. 2003, accepted date : Jul. 2003]