

Bayes Estimation of Stress-Strength System Reliability under Asymmetric Loss Functions

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Abstract

Bayes estimates of reliability for the stress-strength system are obtained with respect to LINEX loss function. A reference prior distribution of the reliability is derived and Bayes estimates of the reliability are also obtained. These Bayes estimates are compared with corresponding estimates under squared-error loss function.

Keywords : Bayes estimator, LINEX loss, stress-strength syetem, reliability

1. Introduction

In the Bayesian estimation of reliability function, symmetric loss function may be inappropriate. Overestimation of reliability function or average failure time is usually much more serious than underestimation. Also, an underestimation of the failure rate results in more serious consequences than an overestimation. For example, in the 1986 disaster of the space shuttle Challenger, NASA underestimated the failure rate and therefore overestimated the reliability of solid-fuel rocket booster(see Feynman, 1987). In dam construction, an underestimation of the peak water level is usually much more serious than an overestimation. For examples of the problem of Bayes estimation with asymmetric loss functions, see Ferguson(1967), Zellner and Geisel(1968), Canfield(1970), Aitchson and Dunsmore(1975), and Zellner(1986). Varian(1975) introduced a very useful asymmetric LINEX loss function that rises approximately exponentially on one side of zero and approximetely linearly on the other side of zero. Varian's LINEX loss function is

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$$L(\Delta) = b \exp(a\Delta) - c\Delta - b, \quad (1a)$$

where Δ denotes the estimation error, and a , $c \neq 0$ and $b > 0$ are constants. Enis and Geisser(1971) obtained a Bayes estimator of R in the exponential case. Gupta and Gupta(1988) obtained a Bayes estimator of R in the case of different location parameters and a common scale parameter under symmetric loss function.

In this paper Bayes estimators of $R = P(X < Y)$ are obtained for two independent exponential distributions. In Section 2, LINEX loss functions are introduced. In Section 3, a reference prior distribution is derived and Bayes estimators of R are obtained under LINEX loss function.

2. LINEX Loss Functions

Let $\Delta = \hat{\theta} - \theta$ denote the estimation error in using $\hat{\theta}$ to estimate θ . From (1a), it is clear that $L(0) = 0$. Also, for a minimum to exist at $\Delta = 0$, we must have $ab = c$, and thus, (1a) can be rewritten as

$$L(\Delta) = b(\exp(a\Delta) - a\Delta - 1), \quad (1b)$$

where $a \neq 0$ is a shape parameter, and $b > 0$ is a scale parameter. The sign of the shape parameter a reflects the direction of the asymmetry, $a > 0$ ($a < 0$) if overestimation is more (less) serious than underestimation; and the magnitude of a reflects the degree of asymmetry. For small values of $|a|$, the loss function is almost symmetric and not far from a squared error loss function.

Indeed, on expanding $e^{a\Delta} \simeq 1 + a\Delta + a^2\Delta^2/2$, $L(\Delta) = a^2\Delta^2/2$, a squared error loss function. However, when $|a|$ assumes appreciable values, optimal point estimates will be quite different from those obtained with a symmetric squared error loss function. See Varian(1975), Zellner(1986) and Parsian(1990) for more details on point estimation under LINEX loss function.

It is well known that for a squared error loss function, the Bayes estimator is obtained as posterior mean. Let $p(\theta|D)$ denote the posterior density function of θ where D denotes the observed data. Let E_θ denote the posterior expectation with respect to $p(\theta|D)$. The Bayes estimator under LINEX loss function is then given by (see Varian(1975)),

$$\hat{\theta} = -\frac{1}{a} \log E_\theta[\exp(-a\theta)|D]. \quad (2)$$

3. Bayes Estimation of $R = P(X < Y)$

Let X_1, X_2, \dots, X_{n_1} and Y_1, Y_2, \dots, Y_{n_2} be two independent samples from exponential distributions with means θ_1 and θ_2 , respectively. In this section we consider Bayes estimators of $R = \theta_2 / (\theta_1 + \theta_2)$ with under LINEX loss function. Attempts to obtain concise closed-form Bayes estimates for the reliability of exponential stress-strength models have historically been unsuccessful. Typically these estimates involve unresolved integrals or infinite series. Such estimates must be calculated by numerical methods.

The results presented in this section do not include closed-form estimates of R . These results provide posterior densities for R corresponding to inverted gamma, noninformative, and reference priors. Using these posterior densities, and numerical methods, a Bayes estimate of R corresponding to any loss function can be obtained.

3.1 Case of inverted gamma prior

Suppose that given the hyperparameter $\alpha_i, \beta_i, i=1, 2$, expected stress θ_1 and expected strength θ_2 have independent prior distributions $\theta_i \sim IG(\alpha_i, \beta_i)$, where $IG(a, b)$ denotes the the inverted gamma distribution whose probability density function(pdf) is given by

$$g(\theta; a, b) = \frac{b^a}{\Gamma(a)} \theta^{-(a+1)} e^{-b/\theta}, \quad \theta > 0.$$

Then the posterior density of $\psi (= R)$ is given by

$$\pi_c(\psi | s_1, s_2) = \pi(\psi; \alpha_1 + n_1, \beta_1 + s_1, \alpha_2 + n_2, \beta_2 + s_2), \quad (3)$$

where

$$\pi(\psi; a, b, c, d) = \frac{(b/d)^a}{B(a, c)} \cdot \frac{\psi^{a-1}(1-\psi)^{c-1}}{[1 - (1-b/d)\psi]^{a+c}}, \quad 0 < \psi < 1, \quad (4)$$

$s_1 = \sum_{i=1}^{n_1} x_i$ and $s_2 = \sum_{i=1}^{n_2} y_i$, and $B(a, c) = \Gamma(a)\Gamma(c)/\Gamma(a+c)$. See Enis and Geisser(1971). Therefore the Bayes estimate of R under LINEX loss function is given by

$$\widehat{R}_L = -\frac{1}{a} \log \sum_{i=0}^{\infty} \frac{(-a)^i}{\Gamma(i+1)} \cdot \int_0^1 \pi(\psi; \alpha_1 + n_1 + i, \beta_1 + s_1, \alpha_2 + n_2, \beta_2 + s_2) d\psi. \quad (5)$$

3.2 Case of noninformative prior

The posterior density of ψ with respect to noninformative prior is

$$\pi_f(\psi | s_1, s_2) = \pi(\psi; n_1, s_1, n_2, s_2), \quad (6)$$

which can be obtained by putting $\alpha_i = \beta_i = 0$ in (3), where $\pi(\cdot)$ is defined in (4). Appendix A also shows Jeffreys approach for obtaining noninformative prior.

3.3 Case of reference prior given a nuisance parameter

Marginalization paradoxes such as those of Dawid *et al.*(1973) suggest that any prior obtained for simultaneous estimation about several parameters is in general inappropriate whenever only one of these parameters is of interest. A procedure proposed by Bernardo(1979) and refined by Berger and Bernardo(1989, 1992) is employed to derive a posterior for $\phi = \theta_2/(\theta_1 + \theta_2)$ and $\omega = \theta_1 + \theta_2$ when ω is regarded as a nuisance parameter.

Theorem 1

The posterior density of ϕ with respect to reference prior is given by

$$\pi_B(\phi | D) = \pi(\phi, n_1, s_1, n_2, s_2). \quad (7)$$

Proof. See Appendix B.

We note that reference prior corresponds to the noninformative prior. It is known that the posterior (7) does not depend on the form chosen for the nuisance parameters. See Bernardo and Smith(1994, p.325).

Example

Independent random samples of stresses(X) and strengths(Y) of sizes $n_1 = 8$ and $n_2 = 7$ were generated from two exponential distributions with $E[X] = \theta_1 = 1.0$ and $E[Y] = \theta_2 = 4.0$. These are:

Stresses : 0.034, 1.344, 0.266, 0.563, 0.168, 3.117, 0.508, and 0.965,

Strengths : 1.525, 0.328, 2.896, 12.409, 1.597, 2.147, and 5.332.

These yield $R = 0.8$, $s_1 = \sum_{i=1}^{n_1} x_i = 6.965$ and $s_2 = \sum_{i=1}^{n_2} y_i = 26.234$, and the posterior density with respect to noninformative prior

$$\pi_f(\phi | s_1, s_2) = \frac{0.5931\phi^7(1-\phi)^6}{(1-0.7345\phi)^{15}}, 0 < \phi < 1. \quad (8)$$

If priors about θ_1 and θ_2 are $IG(3,2)$ and $IG(3,6)$ then the marginal prior density of ϕ is

$$\pi(\phi) = \frac{1.1111\phi^2(1-\phi)^2}{(1-0.6667\phi)^6}, 0 < \phi < 1, \quad (9)$$

and the posterior density is

$$\pi_c(\psi | s_1, s_2) = \frac{1.4230\psi^{10}(1-\psi)^9}{(1-0.7219\psi)^{21}}, 0 < \psi < 1. \quad (10)$$

Graphs of (8), (9), and (10) are displayed in Figure 1. Table 1 lists Bayes estimates of R obtained from these densities under squared-error loss and LINEX loss. From Table 1, as expected, the Bayes estimates of R decrease as a increase. The estimate obtained with respect to (8) is found to be smaller than the estimates with respect to (9) and (10).

Table 1. Bayes Estimates of R under squared error loss and LINEX loss

a	$\pi(\psi)$	$\pi_c(\psi s_1, s_2)$	$\pi_f(\psi s_1, s_2)$
-2	0.9550	0.8858	0.8253
-1	0.9245	0.8762	0.8099
0	0.8986	0.8668	0.7964
1	0.8734	0.8575	0.7831
2	0.8455	0.8482	0.7685

(Note: For $a = 0$, LINEX loss and squared error loss are identical)

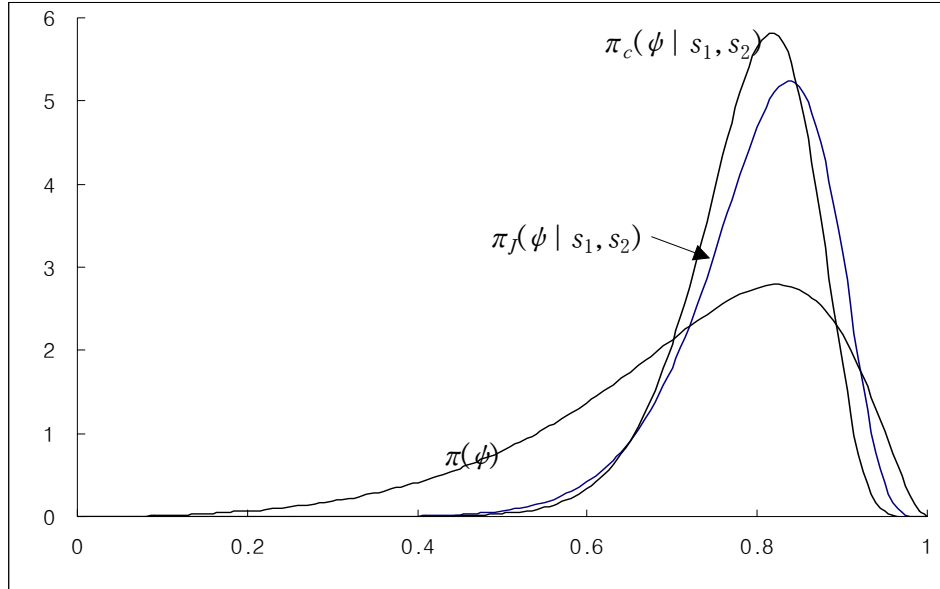


Figure 1. Prior and Posterior Densities of ψ

Appendix A : Derivation of Jeffreys Prior

The likelihood function of (θ_1, θ_2) is

$$l(\theta_1, \theta_2 | S_1, S_2) \propto \frac{\exp(-(S_1/\theta_1 + S_2/\theta_2))}{\theta_1^{n_1} \theta_2^{n_2}}, \quad \theta_i > 0,$$

from which the likelihood function of $\psi = \theta_2/(\theta_1 + \theta_2)$ and $\omega = \theta_1 + \theta_2$ is given by

$$l(\psi, \omega | S_1, S_2) \propto \frac{\exp(-(S_1/((1-\psi)\omega) + S_2/\psi))}{\psi^{n_2} (1-\psi)^{n_1} \omega^n}. \quad (\text{A.1})$$

The Fisher information matrix for ψ and ω is obtained by taking expectations of the negative of second partial derivatives of the log-likelihood function (A.1). Observe

$$l(\psi, \omega | S_1, S_2) = \text{constant} - n_2 \log \psi - n_1 \log(1-\psi) - n \log \omega - \frac{1}{\omega} \left(\frac{S_1}{(1-\psi)} + \frac{S_2}{\psi} \right).$$

Hence,

$$\begin{aligned} I_{11}(\psi, \omega) &= -E_{\psi, \omega} \left\{ \frac{\partial^2}{\partial \psi^2} \log(\psi, \omega | S_1, S_2) \right\} \\ &= \frac{n_2}{\psi^2} + \frac{n_1}{(1-\psi)^2}, \end{aligned}$$

$$I_{12}(\phi, \omega) = \frac{1}{\omega} \left(\frac{n_2}{\phi} - \frac{n_1}{1-\phi} \right), \text{ and } I_{22}(\phi, \omega) = \frac{n}{\omega^2}.$$

The Fisher information matrix for ϕ and ω is then

$$I(\phi, \omega) = (I_{ij})(\phi, \omega) = \begin{bmatrix} \frac{n_2}{\phi^2} + \frac{n_1}{(1-\phi)^2} & \frac{1}{\omega} \left(\frac{n_2}{\phi} - \frac{n_1}{1-\phi} \right) \\ \frac{1}{\omega} \left(\frac{n_2}{\phi} - \frac{n_1}{1-\phi} \right) & \frac{n}{\omega^2} \end{bmatrix}, \quad (\text{A.2})$$

and therefore,

$$|I(\phi, \omega)| = \frac{n_1 n_2}{\phi^2 (1-\phi)^2 \omega^2},$$

where $|I|$ denotes the determinant of the matrix I . Hence, the Jeffreys prior for (ϕ, ω) is

$$\begin{aligned} \pi(\phi, \omega) &\propto \{|I(\phi, \omega)|\}^{1/2} \\ &\propto \frac{1}{\phi(1-\phi)\omega}, \quad 0 < \phi < 1, \quad \omega > 0. \end{aligned}$$

Appendix B : Proof of Theorem 1

The procedure, described in Berger and Bernardo(1989), and Bernardo and Smith(1994, p.326) proceeds as follows:

1. Let $\pi(\omega|\phi)$ be the Jeffreys prior for ω with ϕ given, defined by $\pi(\omega|\phi) \propto (I_{22}(\phi, \omega))^{1/2}$. Observe, from (A.2),

$$I_{22}(\phi, \omega) = \frac{n}{\omega^2}.$$

This implies that

$$\pi(\omega|\phi) \propto \frac{1}{\omega}, \quad \omega > 0.$$

2. Choose an increasing sequence $\{\Lambda_i\}$ of subsets of the parameter space Λ for (ϕ, ω) , such that $\bigcup_i \Lambda_i = \Lambda$ and $\pi(\omega|\phi)$ has finite mass on $\Omega_i(\omega) = \{\omega; (\phi, \omega) \in \Lambda_i\}$ for all ϕ . Then normalize $\pi_i(\omega|\phi)$ on each $\Omega_i(\omega)$, obtaining $\pi_i(\omega|\phi) = c_i(\phi) \cdot \pi(\omega|\phi) \cdot \mathbb{I}[\omega \in \Omega_i(\omega)]$, where $c_i(\phi) = 1 / \int_{\Omega_i(\omega)} \pi(\omega|\phi) d\omega$ and $\mathbb{I}[\cdot]$ is denotes an usual indicator function. Let $\Lambda = (0, 1) \times (0, \infty)$ and $\Lambda_i = (0, 1) \times (1/i, i)$. Let $\Omega_i(\omega) = \{\omega; 1/i < \omega < i\}$, $i = 1, 2, \dots$, be a suitable sequence of subsets of Λ not depending on ϕ , over which $\pi(\omega|\phi)$ can be normalized. It follows that

$$c_i(\phi) = 1 / \int_{1/i}^i 1/\omega d\omega = \frac{1}{2 \log i},$$

and

$$\pi_i(\omega|\phi) = \frac{1}{2\omega \log i}, \quad 1/i < \omega < i.$$

3. Find the marginal reference prior for ϕ with respect to $\pi_i(\omega|\phi)$. This is

$$\pi_i(\phi) \propto \exp \left\{ \int_{\Omega_i(\omega)} \pi_i(\omega|\phi) \cdot \log \left[\frac{|I(\phi, \omega)|}{I_{22}(\phi, \omega)} \right] d\omega \right\}.$$

Assuming the integral exists,

$$\begin{aligned} \pi_i(\phi) &\propto \exp \left\{ \int_{1/i}^i \frac{1}{2\omega \log i} \log \left[\frac{\omega^2}{\omega^2 \phi^2 (1-\phi)^2} \right]^{1/2} d\omega \right\} \\ &= \frac{1}{\phi(1-\phi)}. \end{aligned}$$

4. Define the prior for (ϕ, ω) when ω is a nuisance parameter by

$$\pi(\phi, \omega) \propto \pi(\omega|\phi) \cdot \lim_{i \rightarrow \infty} \left\{ \frac{c_i(\phi) \pi_i(\phi)}{c_i(\phi_0) \pi_i(\phi_0)} \right\},$$

where ϕ_0 is any fixed point. Assuming the limit exists

$$\begin{aligned} \pi(\phi, \omega) &\propto \frac{1}{\omega} \cdot \lim_{i \rightarrow \infty} \left\{ \frac{2(\log i) \phi_0 (1-\phi_0)}{2(\log i) \phi (1-\phi)} \right\} \\ &\propto \frac{1}{\phi(1-\phi)\omega}, \quad 0 < \phi < 1, \quad \omega > 0. \end{aligned}$$

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