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Multi-variate Fuzzy Polynomial Regression using Shape Preserving Operations[1]

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Abstract

In this paper, we prove that multi-variate fuzzy polynomials are universal approximators for multi-variate fuzzy functions which are the extension principle of continuous real-valued function under T_W-based fuzzy arithmetic operations for a distance measure that Buckley et al.(1999) used. We also consider a class of fuzzy polynomial regression model. A mixed non-linear programming approach is used to derive the satisfying solution.

1. Introduction

For many years statistical linear regression has been used in almost every field of science. The purpose of regression analysis is to explain the variation of a dependent variable Y in terms of the variation of explanatory variables X as $Y = f(X)$ where $f(X)$ is a linear function. The use of statistical linear regression is bounded by some strict assumptions about the given data, that is, the unobserved error term are mutually independent and identically distributed. As a result, the statistical regression model can be applied only if the given data are distributed according to a statistical model, and the relation between x and y is crisp.

Since Tanaka et al. (1982) proposed a study in linear regression analysis with fuzzy model, the fuzzy regression analysis has been widely studied and applied in a variety of substantive areas. A collection of recent papers dealing with several

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approaches to fuzzy regression analysis can be found in Kacprzk and Fedrizzi (1992).

Recently, Buckley, Feuring and Hayashi (1999) argued that a very impact class of fuzzy functions in multi-variate non-linear fuzzy regression is the multi-variate fuzzy polynomials using a distance measure on the collection of fuzzy numbers under "min"-norm based fuzzy arithmatic operations. And they introduced an evolution algorithm which search library of multi-variate fuzzy polynomials for the one that best fits some data, generated by a multi-variate fuzzy function.

Recently, Hong et al. (2001a, 2001b, 2001c) presented a new method to evaluate fuzzy linear and non-linear regression models distance where both input data and output data are fuzzy numbers, using shape preserving fuzzy arithmetic operations.

Since T_W -based fuzzy arithmetic operations preserves the shape of fuzzy numbers under addition and multiplication, it simplifies the computation of fuzzy arithmetic operations.

In this paper, we prove that multi-variate fuzzy polynomials are universal approximators for multi-variate fuzzy functions which are the extension principle extension of continuous real-valued function under T_W-based fuzzy arithmetic operations for a distance measure that Buckley et al. (1999) used. We also consider fuzzy quadratic polynomial regression for least-square fitting using the distance measure that Buckley et al. (1999) used. This problem is mixed nonlinear programming problem. We derive the solution using general non-linear programming problem.

2. Preliminaries

A fuzzy number is a convex subset of the real line R with a normalized membership function. A triangular fuzzy number $\tilde{a}$ denoted by (a, α, β) is defined as

$$\tilde{a}(t) = \begin{cases} 1 - \frac{|a-t|}{\alpha} & \text{if} \quad a-\alpha \le t \le a, \\ 1 - \frac{|a-t|}{\beta} & \text{if} \quad a \le t \le a+\beta, \\ 0 & \text{otherwise}, \end{cases}$$

where $a \in R$ is the center and $\alpha > 0$ is the left spread, $\beta > 0$ is the right spread of $\tilde{a}$.

If $\alpha = \beta$, then the triangular fuzzy number is called a symmetric triangular fuzzy number and denoted by (a, α).

A $L-R$ fuzzy number $\tilde{a} = (a, \alpha, \beta)_{LR}$ is a function from the reals into the interval $[0, 1]$ satisfying

$$\tilde{a}(t)=\begin{cases} R\left(\frac{t-a}{\beta}\right) & \text{for} \quad a\le t\le a+\beta, \\ L\left(\frac{a-t}{\alpha}\right) & \text{for} \quad a-\alpha\le t\le a, \\ 0 & \text{else}, \end{cases}$$

where L and R are non-decreasing and continuous functions from $[0,1]$ to $[0,1]$ satisfying $L(0)=R(0)=1$ and $L(1)=R(1)=0$. If $L=R$ and $\alpha=\beta$, then the symmetric $L-L$ fuzzy number is denoted $(a,\alpha)_L$.

An α-cut of a fuzzy number $\tilde{A}$, written as $[\tilde{A}]\alpha$, is defined as $\{x\,|\tilde{A}(x)\ge\alpha\}$, for $0\le\alpha\le1$.

Now, we may present the (restricted) fuzzy regression problem. The extension to $\tilde{X}_i$ a fuzzy vector, for the linear case, is straightforward. So, for now we consider $\tilde{X}_i$ a single fuzzy number.

Let $\mathcal{F}$ denote all fuzzy numbers and let $\mathcal{F}_{LR}$ be $L-R$ fuzzy numbers. A function mapping $\mathcal{F}_{LR}$ into $\mathcal{F}$ will be written as $F(\tilde{X}\,;\,\tilde{K}_1,\cdots,\,\tilde{K}_n)$ where $\tilde{X}$ is the variable in $\mathcal{F}_{LR}$ and the $\tilde{K}$ are parameters (constants) also in $\mathcal{F}_{LR}$. For example, $F(\tilde{X}\,;\,\tilde{K}_1,\,\tilde{K}_2)=\tilde{K}_2\tilde{X}+\tilde{K}_1$, a fuzzy linear function, is one of these functions.

Let Ω be some fixed collection of $F(\tilde{X}\,;\,\tilde{K}_1,\cdots,\,\tilde{K}_n)$ mapping $\mathcal{F}_{LR}$ into $\mathcal{F}$. For example, Ω could be all fuzzy linear functions, or all fuzzy polynomial functions of degree less than four.

Let $(\tilde{X}_i,\,\tilde{Z}_i)$, $1\le i\le p$, be some data $\tilde{X}_i$ in $\mathcal{F}_{LR}$ and $\tilde{Z}_i$ in $\mathcal{F}$. The fuzzy regression problem is to find F in Ω that "best" explains this data. For any F in Ω let $\tilde{Y}_i=F(\tilde{X}\,;\,\tilde{K}_1,\cdots,\,\tilde{K}_n)$, $1\le i\le p$, and let D be a metric on the collection of fuzzy numbers. We measure "best" through the error function.

$$E(F)=\frac{1}{p}\sum_{i=1}^{p}D^2(\tilde{Z}_i,\,\tilde{Y}_i), \tag{1}$$

where $\tilde{Y}_i=F(\tilde{X}\,;\,\tilde{K}_1,\cdots,\,\tilde{K}_n)$. The (restricted) fuzzy regression problem based on Ω is to find F^* in Ω so that

$$\inf_{F\in\Omega}(E(F))=E(F^*). \tag{2}$$

If the problem in Eq. (2) has a solution F^* we will say that F^* best explains the data with respect to Ω.

The Section 4 discusses polynomial types of fuzzy functions we will place into Ω.

The metric we will use is (Buckely, Feuring and Hayashi(1999)):

$$D(\tilde{M}, \tilde{N}) = \sup_{\alpha} H([\tilde{M}]\alpha, [\tilde{N}]\alpha), \tag{3}$$

where H is the Hausdorff distance between nonempty subsets of the reals and $\tilde{N}$, $\tilde{M}$ are two fuzzy numbers. Since α-cuts of fuzzy numbers are always closed, bounded intervals, we get

$$D(\tilde{M}, \tilde{N}) = \sup_{\alpha} \max\{|m_1(\alpha) - n_1(\alpha)|, |m_2(\alpha) - n_2(\alpha)|\} \tag{4}$$

where $[\tilde{M}]\alpha = [m_1(\alpha), m_2(\alpha)]$ and $[\tilde{N}]\alpha = [n_1(\alpha), n_2(\alpha)]$, for all α.

It is noted that for $\tilde{A}_1 = (a_1, \alpha_1, \beta_1)_{LR}$, $\tilde{A}_2 = (a_2, \alpha_2, \beta_2)_{LR}$ we have

$$D(\tilde{A}_1, \tilde{A}_2) = \max\{|a_1 - a_2|, |(a_1 - \alpha_1) - (a_2 - \alpha_2)|, |(a_1 + \beta_1) - (a_2 + \beta_2)|\}. \tag{5}$$

For simlicity, we are only considering $L-R$ fuzzy numbers in $\mathcal{F}_{LR}$.

A binary operation T on the unit interval is said to be triangular norm(t-norm for short) iff T is associative, commutative, non-decreasing and $T(x,1) = x$ for each $x \in [0,1]$. Moreover, every t-norm satisfies the following inequality,

$$T_W(a, b) \le T(a, b) \le \min(a, b) = T_M(a, b)$$

where,

$$T_W(a, b) = \begin{cases} a & \text{if} \quad b = 1, \\ b & \text{if} \quad a = 1, \\ 0 & \text{otherwise}. \end{cases}$$

The crucial importance of $T_M(a, b)$, $a \cdot b$, $\max(0, a+b-1)$ and $T_W(a, b)$ is emphasized from a mathematical point of view in Ling (1965) among others.

The usual arithmetical operations of real numbers can be extended to the arithmetical operations on fuzzy numbers by means of extension principle of Zadeh (1965) based on a triangular norm T. Let $\tilde{A}$, $\tilde{B}$ be fuzzy numbers of reals line R. The fuzzy number arithmetic operations are summarized as follows:

Fuzzy number addition $\oplus$:

$$(\tilde{A} \oplus \tilde{B})(z) = \sup_{x+y=z} T(\tilde{A}(x), \tilde{B}(y)), \tag{6}$$

Fuzzy number multiplication $\otimes$:

$$(\tilde{A} \otimes \tilde{B})(z) = \sup_{x \cdot y = z} T(\tilde{A}(x), \tilde{B}(y)).$$

The addition(subtraction) rule for $L-R$ fuzzy numbers is well known in the case of T_M-based addition and then the resulting sum is again on $L-R$ fuzzy numbers, i.e., the shape is preserved. Diamond (1988) used T_M-based addition in his paper. It is also known that T_W-based addition preserves the shape of $L-R$ fuzzy numbers (Kolesárová(1995), Mesiar(1997)). In practical computation, it is natural to require the preserving the shape of fuzzy numbers during the multiplication. Of course, we know that T_M-based multiplication does not preserve the shape of $L-R$ fuzzy numbers. But it is known by Hong and Do (1997) that T_W induces shape preserving multiplication of $L-R$ fuzzy numbers. Recently, Hong (2000) showed that T_W is the unique t-norm which induces shape preserving in multiplication of $L-R$ fuzzy numbers.

Hong et al. (2001a, 2001b, 2001c) used T_W -based fuzzy arithmetic operations. Let $\mathcal{A}_i=(a_i,\alpha_i)_L$ and $\mathcal{X}_{ij}=(x_{ij},\gamma_{ij})_L$, $i=1,2,\cdots,n$, $j=1,2,\cdots,p$. Then the membership function of $\mathcal{Y}_i=(\mathcal{A}\otimes \mathcal{X}_{i1})\oplus(\mathcal{A}_2\otimes \mathcal{X}_{i2})\oplus\cdots\oplus(\mathcal{A}_p\otimes \mathcal{X}_{ip})$ is given by

$$\mathcal{Y}_i=(\sum_{j=1}^{p} a_i x_{ij},\ \max{}_{1\le j\le p}(|a_i|\gamma_{ij},|x_{ij}|\alpha_i))_L. \tag{7}$$

Let $\mathcal{B}_i$, $i=1,2,\cdots,n$ be fuzzy number. Define $\sum_{i=1}^{n}\mathcal{B}_i=\mathcal{B}_1\oplus\cdots\oplus\mathcal{B}_n$. A possibilistic quadratic polynomial systems whose parameter is defined as

$$\mathcal{Y}=\sum_{j=1}^{p}(\mathcal{A}_j\otimes \mathcal{X}_j)\oplus\sum_{1\le l\le k\le p}(\mathcal{A}_{l,k}\otimes \mathcal{X}_l\otimes \mathcal{X}_k) \tag{8}$$

where $\mathcal{A}=\{\mathcal{A}_j,\ \mathcal{A}_{l,k}|1\le j\le p, 1\le l\le k\le p\}$ is a fuzzy parameters and $\mathcal{X}=(\mathcal{X}_1,\cdots,\mathcal{X}_p)$ is a fuzzy vector. Using T_W-based arithmetic operations, we have the following lemma by (7).

Proposition 2.1 Let $\mathcal{A}_j=(a_j,\alpha_j)_L$, $\mathcal{A}_{l,k}=(a_{l,k},\alpha_{l,k})_L$ and $\mathcal{X}_j=(x_j,\gamma_j)$. Then the possibilistic quadratic polynomial function with fuzzy parameter $\mathcal{A}_j$, $\mathcal{A}_{l,k}$ and fuzzy variables $\mathcal{X}_j$, $j=1,2,\cdots,p$, $1\le l\le k\le p$ is given by

$$\begin{aligned}\mathcal{Y}=(&\sum_{j=1}^{p} a_j x_j+\sum_{1\le l\le k\le p} a_{l,k}x_l x_k,\\ &\max\{\max{}_{1\le j\le p}(|a_j|\gamma_j,\alpha_j|x_j|),\max{}_{1\le l\le k\le p}(\alpha_{l,k}|x_l||x_k|,|a_{l,k}|\gamma_l|x_k|,|a_{l,k}||x_l|\gamma_k)\})_L.\end{aligned} \tag{9}$$

3. Universal approximator

A function mapping $\mathcal{F}^n_{LR}$ into $\mathcal{F}$ will be written $F(\mathcal{X};\mathcal{K})$ where $\mathcal{X}=(\mathcal{X}_1,\cdots,\mathcal{X}_n)$, $\mathcal{K}=(\mathcal{K}_1,\cdots,\mathcal{K}_n)$, the variables $\mathcal{K}_j$ are also parameters (constants) in $\mathcal{F}_{LR}$. From now on, to simplify the discussion n will be 2 so that all our multi-variate fuzzy functions have only two independent variables.

We obtain such an F via the extension principle. Let $f(x_1,x_2;k_1,\cdots,k_m):[a,b]\times$

$[c, d] \to R$ where the parameters k_i belong to (closed, bounded) intervals I_i, $1 \le i \le m$. Although the k_i are constants we will consider f having $m+2$ variables so it is a continuous mapping from $[a, b] \times [c, d] \times \Pi^n_{i=1} I_i$ into R. Now we extend f, using the extension principle to $F(\tilde{X}_1, \tilde{X}_2; \tilde{K}_1, \cdots, \tilde{K}_m)$ for $\tilde{X}_i$ in $\mathcal{F}_{LR}$, $\tilde{X}_1$ in $[a, b]$, $\tilde{X}_2$ in $[c, d]$ and all the $\tilde{K}_j$ in F_{LR} with $\tilde{K}_j$ in I_j, $1 \le j \le m$. Let $\tilde{Z} = F(\tilde{X}_1, \tilde{X}_2; \tilde{K}_1, \cdots, \tilde{K}_m)$ with $\tilde{Z}$ in $\mathcal{F}$.

We will use the notation $p_\theta(x_1, x_2; k_1, \cdots, , k_m)$ for a polynomial in variables $x_1, x_2, k_1, \cdots, k_m$ of degree d_1 in x_1, d_2 in x_2, d_3 in $k_1, \cdots, d_{m+2}$ in k_m with $\theta = (d_1, d_2; d_3, \cdots, d_{m+2})$. Given ε, there is a p_θ so that (Taylor(1965))

$$| f(x_1, x_2; k_1, \cdots, k_m) - p_\theta(x_1, x_2; k_1, \cdots, k_m) | < \frac{\varepsilon}{(m+2)}, \tag{10}$$

for all $x_1 \in [a, b]$, $x_2 \in [c, d]$, and $k_j \in I_j$, $1 \le j \le m$.

Now we use the extension principle to extend p_θ to $\overline{P}_\theta(\tilde{X}_1, \tilde{X}_2; \tilde{K}_1, \cdots, \tilde{K}_m) = \tilde{Y}$ for $\tilde{X}_1 \in [a, b]$, $\tilde{X}_2 \in [c, d]$, the $\tilde{K}_j \in I_j$ and $\tilde{X}_1, \tilde{X}_2, \tilde{K}_1, \tilde{K}_2, \cdots, \tilde{K}_m$ all in F_{LR}, $\tilde{Y}$ in F.

If $T = T_W$, then by a result of Fuller and Koresztfalvi(1991)

$$\begin{aligned} \tilde{Z}[\alpha] &= \{F(\tilde{X}_1, \tilde{X}_2; \tilde{K}_1, \cdots, \tilde{K}_m) \ge \alpha\} \\ &= f([\tilde{X}_1]^\alpha, [\tilde{X}_2]^1; [\tilde{K}_1]^1, \cdots, [\tilde{K}_m]^1) \\ & \cup f([\tilde{X}_1]^1, [\tilde{X}_2]^\alpha; [\tilde{K}_1]^1, \cdots, [\tilde{K}_m]^1) \\ & \quad \cdots \\ & \cup f([\tilde{X}_1]^1, [\tilde{X}_2]^1; [\tilde{K}_1]^1, \cdots, [\tilde{K}_{m-1}]^1, [\tilde{K}_m]^\alpha) \end{aligned}$$

and similarly

$$\begin{aligned} \tilde{Y}[\alpha] &= \{P_\theta(\tilde{X}_1, \tilde{X}_2; \tilde{K}_1, \cdots, \tilde{K}_m) \ge \alpha\} \\ &= p_\theta([\tilde{X}_1]^\alpha, [\tilde{X}_2]^1; [\tilde{K}_1]^1, \cdots, [\tilde{K}_m]^1) \\ & \quad \cup p_\theta([\tilde{X}_1]^1, [\tilde{X}_2]^\alpha; [\tilde{K}_1]^1, \cdots, [\tilde{K}_m]^1) \\ & \quad \cdots \\ & \quad \cup p_\theta([\tilde{X}_1]^1, [\tilde{X}_2]^1; [\tilde{K}_1]^1, \cdots, [\tilde{K}_{m-1}]^1, [\tilde{K}_m]^\alpha). \end{aligned}$$

Here, we note that

$$
\begin{aligned}
&|\min[\tilde{Z}]\alpha - \min[\tilde{Y}]\alpha| \\
&\quad \le |\min f([\tilde{X}_1]^{\alpha},[\tilde{X}_2]^{1};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_m]^{1}) \\
&\quad - \min p_{\theta}([\tilde{X}_1]^{\alpha},[\tilde{X}_2]^{1};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_m]^{1})| \\
&\quad + |\min f([\tilde{X}_1]^{1},[\tilde{X}_2]^{\alpha};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_m]^{1}) \\
&\quad - \min p_{\theta}([\tilde{X}_1]^{1},[\tilde{X}_2]^{\alpha};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_m]^{1})| \\
&\quad \vdots \\
&\quad + |\min f([\tilde{X}_1]^{1},[\tilde{X}_2]^{1};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_{m-1}]^{1},[\tilde{K}_m]^{\alpha}) \\
&\qquad - \min p_{\theta}([\tilde{X}_1]^{1},[\tilde{X}_2]^{1};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_{m-1}]^{1},[\tilde{K}_m]^{\alpha})|
\end{aligned}
$$

and

$$
\begin{aligned}
&|\max[\tilde{Z}]\alpha - \max[\tilde{Y}]\alpha| \\
&\quad \le |\max f([\tilde{X}_1]^{\alpha},[\tilde{X}_2]^{1};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_m]^{1}) \\
&\quad - \max p_{\theta}([\tilde{X}_1]^{\alpha},[\tilde{X}_2]^{1};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_m]^{1})| \\
&\quad + |\max f([\tilde{X}_1]^{1},[\tilde{X}_2]^{\alpha};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_m]^{1}) \\
&\quad - \max p_{\theta}([\tilde{X}_1]^{1},[\tilde{X}_2]^{\alpha};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_m]^{1})| \\
&\quad \vdots \\
&\quad + |\max f([\tilde{X}_1]^{1},[\tilde{X}_2]^{1};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_{m-1}]^{1},[\tilde{K}_m]^{\alpha}) \\
&\qquad - \max p_{\theta}([\tilde{X}_1]^{1},[\tilde{X}_2]^{1};[\tilde{K}_1]^{1},\cdots,[\tilde{K}_{m-1}]^{1},[\tilde{K}_m]^{\alpha})|.
\end{aligned}
$$

Now, from equation (10), we easily prove that (same ε as in equation (10))

Theorem 1. $D(F(\tilde{X}_1, \tilde{X}_2; \tilde{K}_1, \cdots, \tilde{K}_m), P_{\theta}(\tilde{X}_1, \tilde{X}_2; \tilde{K}_1, \cdots, \tilde{K}_m)) < \varepsilon$, for all $\tilde{X}_1 \in [a, b]$, $\tilde{X}_2 \in [c, d]$ and all $\tilde{K}_j \in I_j$, $1 \le j \le m$.

This means that multi-variate fuzzy polynomials are universal approximators.

4. Fuzzy polynomial regression

In this section, we consider fuzzy quadratic polynomial regression model for least-square fitting with respect to the D-metric.

Let $\mathcal{F}_{LR}(R)$ be the set of all $L-R$ fuzzy numbers. In order to solve fuzzy least squares optimization problem in $\mathcal{F}_{LR}(R)$, we use the metric D which is defined as distance as follows:

$$D(\tilde{A}_1, \tilde{A}_2)^2 = \max((a_1-a_2)^2,\ ((a_1-\alpha_1)-(a_2-\alpha_2))^2,\ ((a_1+\beta_1)-(a_2+\beta_2))^2) \quad (11)$$

where $\tilde{A}_1 = (a_1, \alpha_1, \beta_1)_{LR}$, $\tilde{A}_2 = (a_2, \alpha_2, \beta_2)_{LR}$.

In this section, we consider the following model:

$$(\mathrm{P}):\ \tilde{Y} = \sum_{j=1}^{p}(\tilde{A}_j \otimes \tilde{X}_j) \oplus \sum_{1 \le l \le k \le p}(\tilde{A}_{l,k} \otimes \tilde{X}_l \otimes \tilde{X}_k) \quad (12)$$

where $\tilde{A}_j$, $\tilde{A}_{l,k}$, $\tilde{X}_j \in \mathcal{F}_{LR}(R)$, $1 \le j \le p$, $1 \le l \le k \le p$.

We assume, throughout this section, that $\tilde{A}_j$, $\tilde{A}_{ij}$, $\tilde{X}$, $\tilde{Y} \in \mathcal{F}_{LR}(R)$ are symmetric $L-R$ fuzzy numbers for computational simplicity. Suppose that observations consist of data pairs $(\tilde{X}_i, \tilde{Y}_i)$, $i=1,2,\cdots,n$, where $\tilde{X}_i = (\tilde{X}_{i1},\cdots,\tilde{X}_{ip})$, $\tilde{X}_{ij}=(x_{ij},\gamma_{ij})_L$, $j=1,\cdots,p$, $\tilde{Y}_i=(y_i,\eta_i)_L$. Each is to be fitted to the data in the sense of best fit with respect to the D_{LR}-metric. In association with the model (P), consider the least-squares optimization problem

$$(D): \text{ Minimize } \quad nE(F) = r(a,\alpha) = \sum_{i=1}^{n} D^2\Big(\sum_{j=1}^{p}(\tilde{A}_{ij}\otimes \tilde{X}_{ij}) \oplus \sum_{1\le l\le k\le p}(\tilde{A}_{l,k}\otimes \tilde{X}_{il}\otimes \tilde{X}_{ik}),\ \tilde{Y}_i\Big) \tag{13}$$

Let $\tilde{A}_j=(a_j,\alpha_j)_L$, and $\tilde{A}_{l,k}=(a_{l,k},\alpha_{l,k})_L$, then by (9)

$$\begin{aligned} nE(f) = &\sum_{i=1}^{n} D^2\Big(\Big(\sum_{j=1}^{p} a_j x_{ij} + \sum_{1\le l\le k\le p} a_{l,k} x_{il} x_{ik},\\ &\max\{\max_{1\le j\le p}(|a_j|\gamma_{ij}, \alpha_j|x_{ij}|),\\ &\max_{1\le l\le k\le p}(\alpha_{l,k}|x_{il}||x_{ik}|, |a_{l,k}|\gamma_{il}|x_{ik}|, |a_{l,k}||x_{il}|\gamma_{ik})\}\Big)_L,\ \tilde{Y}_i\Big)\\ = \max &\sum_{i=1}^{n}\Big\{\Big[\sum_{j=1}^{p} a_j x_{ij} + \sum_{1\le l\le k\le p} a_{l,k} x_{il} x_{ik} -\\ &\max\{\max_{1\le j\le p}(|a_j|\gamma_{ij}, \alpha_j|x_{ij}|),\\ &\max_{1\le l\le k\le p}(\alpha_{l,k}|x_{il}||x_{ik}|, |a_{l,k}|\gamma_{il}|x_{ik}|, |a_{l,k}||x_{il}|\gamma_{ik})\} - (y_i-\eta_i)\Big]^2,\\ &\Big[\sum_{j=1}^{p} a_j x_{ij} + \sum_{1\le l\le k\le p} a_{l,k} x_{il} x_{ik} + \max\{\max_{1\le j\le p}(|a_j|\gamma_{ij}, \alpha_j|x_{ij}|),\\ &\max_{1\le l\le k\le p}(\alpha_{l,k}|x_l||x_k|, |a_{l,k}|\gamma_{il}|x_{ik}, |a_{l,k}||x_{il}|\gamma_{ik})\} - (y_i+\eta_i)\Big]^2,\\ &\Big[\sum_{j=1}^{p} a_j x_{ij} + \sum_{1\le l\le k\le p} a_{l,k} x_{il} x_{ik} - y_i\Big]^2\Big\}. \end{aligned}$$

This problem can be computed by mixed QP problem as follows:
Let $M=\{(j,l,k) | 1\le j\le p, 1\le l\le k\le p\}$, and define

$$\begin{aligned} A(i,(j,l,k),H_r) = &\{((a_1,\alpha_1),\cdots,(a_p,\alpha_p),(a_{1,1},\alpha_{1,1}),\cdots,(a_{p,p},\alpha_{p,p}))\in (R^2)^{\frac{p(p+3)}{2}} \mid\\ &\max(|a_j|\gamma_{ij}, |x_{ij}|\alpha_j, \alpha_{l,k}|x_{il}||x_{ik}|, |a_{l,k}|\gamma_{il}|x_{ik}|, |a_{l,k}||x_{il}|\gamma_{ik})\\ &= H_r\} \end{aligned}$$

where $H_1 = a_j\gamma_{ij}$, $a_j\ge 0$, $H_2=-a_j\gamma_{ij}$, $a_j<0$, $H_3=|x_{ij}|\alpha_j$, $H_4=\alpha_{l,k}|x_{il}||x_{ik}|$, $H_5=a_{l,k}\gamma_{il}$ $|x_{ik}|$, $a_{l,k}\ge 0$, $H_6=-a_{l,k}\gamma_{il}|x_{ik}|$, $a_{l,k}<0$, $H_7=a_{l,k}|x_{il}|\gamma_{ik}$, $a_{l,k}\ge 0$, $H_8=-a_{l,k}|x_{il}|\gamma_{ik}$, $a_{l,k}<0$.

Let f and g be functions such that

$$f: \{1,2,\cdots,n\} \Rightarrow M,$$
$$g: M \Rightarrow \{H_1,H_2,\cdots,H_8\}.$$

On $\bigcap_{i=1}^{n} A(i, f(i), g(f(i)))$, (13) is an QP problem and

$$\text{Min } r(a, \alpha) = \text{Min}_{f,g} \text{ Min}_{(a,\alpha)\in \bigcap_{i=1}^{n} A(i,f(i),g(f(i)))} r(a, \alpha).$$

For example, let $n=2$, $p=2$ in (13). Then the model can be written as

$$Y_{i^*} = (A_1^* \otimes X_{i1}) \oplus (A_2^* \otimes X_{i1}) \oplus (A^*_{1,1} \otimes X_{i1} \otimes X_{i2}) \oplus (A^*_{1,2} \otimes X_{i1} \otimes X_{i2}) \oplus (A^*_{2,2} \otimes X_{i2} \otimes X_{i2})$$

and $M = \{(1,1,1), (1,1,2), (1,2,2), (2,1,1), (2,1,2), (2,2,2)\}$. Let f: $\{1,2\} \to M$ be such that $f(1) = (1,1,2)$, $f(2) = (2,1,1)$ and let g: $M \to \{H_1, H_1, \cdots, H_8\}$ be such that $g(f(1)) = g((1,1,2)) = H_5$, $g(f(2)) = g((2,1,1)) = H_3$ Then, on $A(1, f(1), g(f(1))$ $\cap A(2, f(2),\ g(f(2))$, (13) is written as

$$\begin{aligned}
\text{Minimize} \quad r(a,\alpha) = \max\Big\{ & [(\sum_{j=1}^{2} a_j x_{1j} + \sum_{1\le l\le k\le 2} a_{l,k} x_{1l} x_{1k}) - a_{1,2}\gamma_{11}|x_{12}| - (y_1 - \eta_1)]^2, \\
& [(\sum_{j=1}^{2} a_j x_{1j} + \sum_{1\le l\le k\le 2} a_{l,k} x_{1l} x_{1k}) + a_{1,2}\gamma_{11}|x_{12}| - (y_1 + \eta_1)]^2, \\
& [(\sum_{j=1}^{2} a_j x_{1j} + \sum_{1\le l\le k\le 2} a_{l,k} x_{1l} x_{1k}) - y_1)]^2 \Big\} \\
+ \max\Big\{ & [(\sum_{j=1}^{2} a_j x_{2j} + \sum_{1\le l\le k\le 2} a_{l,k} x_{2l} x_{2k}) - |x_{22}|\alpha_2 - (y_2 - \eta_2)]^2 \\
& [(\sum_{j=1}^{2} a_j x_{2j} + \sum_{1\le l\le k\le 2} a_{l,k} x_{2l} x_{2k}) + |x_{22}|\alpha_2 - (y_2 + \eta_2)]^2, \\
& [(\sum_{j=1}^{2} a_j x_{2j} + \sum_{1\le l\le k\le 2} a_{l,k} x_{2l} x_{2k}) - y_2)]^2 \Big\},
\end{aligned}$$

$$\begin{aligned}
& |a_1|\gamma_{11} \le a_{1,2}\gamma_{11}|x_{12}|,\ |x_{11}|\alpha_1 \le a_{1,2}\gamma_{11}|x_{12}|, \\
& \alpha_{1,2}|x_{11}||x_{12}| \le a_{1,2}\gamma_{11}|x_{12}|, \\
& a_{1,2} \ge 0,\ a_{1,2}|x_{11}|\gamma_{12} \le a_{1,2}\gamma_{11}|x_{12}|, \\
& |a_2|\gamma_{22} \le |x_{22}|\alpha_2,\ \alpha_{1,1}|x_{21}||x_{21}| \le |x_{22}|\alpha_2,\ |a_{1,1}||x_{21}|\gamma_{21} \le |x_{22}|\alpha_2
\end{aligned}$$

which is a mixed QP problem with respect to a_j, α_j, $j = 1, 2$, $a_{l,k}$, $\alpha_{l,k}$, $1 \le l \le k \le 2$.

Now we consider all such functions f and g and take minimum with regard to them.

Then we get the desired solution.

Example. We consider the same artificial data shown in Table 1 in Hong and Do (2001a).

Table 1. Fuzzy Input-Output Data of Nonlinear Type

Sample number i	$\tilde{X}_i=(x_i, \gamma_i)$	$\tilde{Y}_i=(y_i, \eta_i)$	Sample number i	$\tilde{X}_i=(x_i, \gamma_i)$	$\tilde{Y}_i=(y_i, \eta_i)$
1	(1.0, 0.5)	(6.3, 2.0)	11	(6.5, 1.5)	(39.9, 3.0)
2	(1.5, 0.5)	(11.1, 1.5)	12	(7.0, 2.5)	(42.0, 1.5)
3	(2.0, 1.0)	(20.0, 2.0)	13	(8.0, 2.0)	(46.1, 2.0)
4	(3.0, 1.0)	(24.0, 1.5)	14	(9.0, 3.0)	(53.1, 4.0)
5	(4.0, 1.0)	(26.1, 1.0)	15	(10.0, 2.0)	(52.0, 5.0)
6	(4.5, 0.5)	(30.0, 3.0)	16	(11.0, 2.0)	(52.5, 3.5)
7	(5.0, 1.5)	(33.8, 2.5)	17	(12.0, 1.0)	(48.0, 3.0)
8	(5.5, 1.0)	(34.0, 3.0)	18	(13.0, 1.0)	(42.8, 2.5)
9	(6.0, 2.0)	(38.1, 2.5)	19	(14.0, 1.0)	(27.8, 2.0)
10	(15.0, 1.0)	(21.9, 1.5)			

Noting that $\tilde{Y}_i=(y_i, \eta_i)$

$$\tilde{A}_0 \oplus (\tilde{A}_1 \otimes \tilde{X}_i) \oplus (\tilde{A}_2 \otimes \tilde{X}_i^2) = (a_0 + a_1x_i + a_2x_i^2, \max(\alpha_0, |a_1|\gamma_i, \alpha_1 x_i, |a_2|x_i\gamma_i, \alpha_2 x_i^2))$$

we minimize

$$\begin{aligned} r(a, \alpha) = & \sum_{i=1}^{19} \max\{([(a_0 + a_1x_i + a_2x_i^2) - \\ & \max(\alpha_0, |a_1|\gamma_i, \alpha_1x_i, |a_2|x_i\gamma_i, \alpha_2x_i^2) - (y_i - \eta_i)]^2, \\ & [(a_0 + a_1x_i + a_2x_i^2) + \\ & \max(\alpha_0, |a_1|\gamma_i, \alpha_1x_i, |a_2|x_i\gamma_i, \alpha_2x_i^2) - (y_i + \eta_i)]^2, \\ & [(a_0 + a_1x_i + a_2x_i^2) - y_i]^2)\}. \end{aligned}$$

Then the solution is

$\tilde{A}_0^*$ =(15.9, 0.0~0.05), $\tilde{A}_1^*$ = (3.19, 0.0~0.26), $\tilde{A}_2^*$ =(-0.13, 0.0~0.01) with $r(a^*, \alpha^*)$=(5111.67729).

5. Conclusion

We proved that multivariate fuzzy polynomials are universal approximators for multivariate fuzzy functions which are the extension principle of continuous real valued multivariate functions under T_W-based fuzzy arithmetic operations.

We also suggested fuzzy quadratic polynomial regression for least-square fitting using the distance measure that Buckley et al.(1999) used under shape preserving operations. We use general mixed nonlinear programming problem to derive the optimal solutions. An artificial example is given.

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