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System Reliability From Stress-Strength Relationship in Bivariate Pareto Distribution

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Abstract

In this paper, We assume that strengths of two components system follow a bivariate pareto distribution. And these two components are subjected to a common stress which is independent of the strength of the components. We obtain maximum likelihood estimator(MLE) for the system reliability from stress-strength relationship. Also we derive asymptotic properties of the MLE and present a numerical study.

Key Words : Bivariate pareto distribution; Common stress; Maximum likelihood estimator; Stress-strength; System reliability.

1. Introduction

In recent years, bivariate pareto distributions have been proposed in the modelling of life times of two-component systems working in a changing environment. Lindley and Singpurwalla(1986) considered the distribution of life lengths measured in a laboratory environment as independent exponential distributions proved that, when they work in a different environment which may be harsher, the same or gentler than the original, the resulting density of life lengths has a bivariate pareto distribution. However, the assumption of independence is unrealistic as in many systems the component life lengths have a

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well-defined dependence structure. With this assumption Bandyapadhyay and Basu(1990), and Veenus and Nair(1994) obtained the bivariate pareto distributions corresponding to some well-known bivariate exponential distributions. Jeevanand(1997) obtained Bayes estimation of the reliability of stress-strength model in bivariate pareto distribution. Hanagal(1996) introduced a new multivariate pareto distribution with interesting properties.

In this paper, we consider two-component system. We assume that strengths of two components system follow a bivariate pareto distribution. And these two components are subjected to a common stress which is independent of the strength of the components. From stress-strength relationship, we obtain MLE for the system reliability and derive asymptotic properties of the MLE. Also we present a numerical example by giving a data set which is generated by computer.

2. Estimation of System Reliability

Let (X_1, X_2) be strengths of two components that follow a bivariate pareto(BVP) distribution. Then the joint survival function of (X_1, X_2) is given by

$$\overline{F}(x_1, x_2) = P(X_1 > x_1\ ,\ X_2 > x_2)$$

$$= \left(\frac{x_1}{\beta}\right)^{-\lambda_1} \cdot \left(\frac{x_2}{\beta}\right)^{-\lambda_2} \cdot max\left(\frac{x_1}{\beta}, \frac{x_2}{\beta}\right)^{-\lambda_3}, \quad \beta \le \min(x_1, x_2) < \infty, \tag{2.1}$$

where $\lambda_1, \lambda_2, \lambda_3 > 0$.

The above BVP model is not absolutely continuous with respect to Lebesgue measure on R^2. That is, in the present model there is provision for simultaneous failure of the components. The marginals of X_i , $i=1,2$ are given by

$$\overline{F}(x_i) = P(X_i > x_i) = \left(\frac{x_i}{\beta}\right)^{-(\lambda_i + \lambda_3)}, \quad i=1,2, \tag{2.2}$$

which are the survival functions of pareto with parameters $(\lambda_i + \lambda_3,\ \beta)$, $i=1,2$.

From (2.1) and (2.2), we can see that the random variables X_1 and X_2 are independent if and only if $\lambda_3 = 0$. And X_1 and X_2 are identically distributed if and only if $\lambda_1 = \lambda_2$. The probability that X_1 and X_2 are equal to each other is $P[X_1 = X_2] = \lambda_3/\lambda$, where $\lambda = \lambda_1 + \lambda_2 + \lambda_3$.

We assume $\beta=1$ in BVP model, the joint survival function of (X_1, X_2) is given by

$$\overline{F}(x_1, x_2) = x_1^{-\lambda_1} \cdot x_2^{-\lambda_2} \cdot (max(x_1, x_2))^{-\lambda_3}. \qquad (2.3)$$

We call the survival function (2.3) as BVP type 2 and the survival function (2.1) as BVP type 1.

Let n_1 be the number of observations with $x_{1i} < x_{2i}$ in the sample, and let n_2 be the number of observations with $x_{2i} < x_{1i}$ in the sample, and let n_3 be the number of observations with $x_{1i} = x_{2i}$ in the sample. Then the likelihood function of the sample of size n is

$$L = \lambda_1^{n_1} \cdot \lambda_2^{n_2} \cdot \lambda_3^{n_3} \cdot (\lambda_1+\lambda_3)^{n_2} \cdot (\lambda_2+\lambda_3)^{n_1} \cdot \beta^{n\lambda} \cdot \left[\prod_{i=1}^{n} x_{1i}\right]^{-(\lambda_1+1)}$$

$$\left[\prod_{i=1}^{n} x_{2i}\right]^{-(\lambda_2+1)} \cdot \left[\prod_{i=1}^{n} max(x_{1i}, x_{2i})\right]^{-\lambda_3} \cdot \left[\prod_{\{i \mid x_{1i} = x_{2i}\}} x_i\right]^{-1} \qquad (2.4)$$

Let Y be the common random stress that follow pareto distribution with parameter (μ, β), that is, distribution function $G(y) = 1 - \left(\frac{y}{\beta}\right)^{-\mu}$ and that Y is independent on (X_1, X_2). Then the reliability of the system reliability from stress-strength relationship is given by

$$R = P[\, Y < max(X_1, X_2)]. \qquad (2.5)$$

For $z > \beta$, the distribution of $Z = max(X_1, X_2)$ is given by

$$\begin{aligned} H(Z) &= P[Z < z] \\ &= P[X_1 < z, \ X_2 < z] \\ &= 1 - \beta^{(\lambda_1+\lambda_3)} \cdot z^{-(\lambda_1+\lambda_3)} - \beta^{(\lambda_2+\lambda_3)} \cdot z^{-(\lambda_2+\lambda_3)} + \beta^{\lambda} \cdot z^{-\lambda}. \end{aligned} \qquad (2.6)$$

Hence the survival function of Z is

$$\begin{aligned} \overline{H}(Z) &= P[Z > z] \\ &= \beta^{(\lambda_1+\lambda_3)} \cdot z^{-(\lambda_1+\lambda_3)} + \beta^{(\lambda_2+\lambda_3)} \cdot z^{-(\lambda_2+\lambda_3)} - \beta^{\lambda} \cdot z^{-\lambda}. \end{aligned} \qquad (2.7)$$

Now, the system reliability(R) is

$$\begin{aligned} R &= P[\, Y < Z] \\ &= \int_{\beta}^{\infty} \overline{H}(y)\, dG(y) \end{aligned}$$

$$= \mu\left[\frac{1}{\lambda_1+\lambda_3+\mu} + \frac{1}{\lambda_2+\lambda_3+\mu} - \frac{1}{\lambda+\mu} \right]. \tag{2.8}$$

In this paper, we focus only on BVP type 2 model. From (2.4), the MLE's of $(\lambda_1, \lambda_2, \lambda_3)$ can be obtained as follows;

$$\frac{n_1}{\lambda_1} + \frac{n_2}{\lambda_1+\lambda_3} - \sum_{i=1}^{n} \log(x_{1i}) = 0, \tag{2.9}$$

$$\frac{n_2}{\lambda_2} + \frac{n_1}{\lambda_2+\lambda_3} - \sum_{i=1}^{n} \log(x_{2i}) = 0, \tag{2.10}$$

$$\frac{n_3}{\lambda_3} + \frac{n_2}{\lambda_1+\lambda_3} + \frac{n_1}{\lambda_2+\lambda_3} - \sum_{i=1}^{n} \log(max(x_{1i}, x_{2i})) = 0. \tag{2.11}$$

By either Newton-Raphson procedure or Fisher's method of scoring, we can easily obtain MLEs of $(\lambda_1, \lambda_2, \lambda_3)$.

Let (X_{1i}, X_{2i}) and Y_i, $i=1,\cdots,n$ be *i.i.d.* random sample of size n and k_1 be the number of observations with $Y_i < max(X_{1i}, X_{2i})$ in the sample. Then the distribution of k_1 is binomial(n, R). Hence, the natural estimate of R is $\hat{R}_N = k_1/n$ which is asymptotic normal distribution with mean R and variance $R(1-R)/n$.

Since the MLE of μ is given by $\hat{\mu} = n/\sum_{i=1}^{n} \log(y_i)$, the estimate of system reliability R in (2.8) based on MLE's of $(\lambda_1, \lambda_2, \lambda_3, \mu)$ is

$$\hat{R}_{MLE} = \hat{\mu}\left[\frac{1}{\hat{\lambda}_1+\hat{\lambda}_3+\hat{\mu}} + \frac{1}{\hat{\lambda}_2+\hat{\lambda}_3+\hat{\mu}} - \frac{1}{\hat{\lambda}+\hat{\mu}} \right], \quad \hat{\lambda} = \hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3. \tag{2.12}$$

By consistency of MLE and delta method, we can see that the asymptotic distribution of $\hat{R}_{MLE}$ is normal distribution with mean R and variance $\Lambda \cdot I^{-1}(\lambda_1, \lambda_2, \lambda_3, \mu) \cdot \Lambda'/n$, where $\Lambda = (\partial R/\partial\lambda_1,\ \partial R/\partial\lambda_2,\ \partial R/\partial\lambda_3,\ \partial R/\mu)$ (See Lehmann(1983), chapter 5).

Here, the elements of Fisher information matrix $I(\lambda_1, \lambda_2, \lambda_3, \mu)$ are given by

$$I_{11} = \frac{1}{\lambda\lambda_1} + \frac{\lambda_2}{\lambda(\lambda_1+\lambda_3)^2}, \quad I_{13} = \frac{\lambda_2}{\lambda(\lambda_1+\lambda_3)^2}, \quad I_{22} = \frac{1}{\lambda\lambda_2} + \frac{\lambda_1}{\lambda(\lambda_2+\lambda_3)^2},$$

$$I_{23}=\frac{\lambda_1}{\lambda(\lambda_2+\lambda_3)^2},\quad I_{33}=\frac{\lambda_1}{\lambda(\lambda_2+\lambda_3)^2}+\frac{\lambda_2}{\lambda(\lambda_1+\lambda_3)^2}+\frac{1}{\lambda\lambda_3},\quad I_{44}=\frac{1}{\mu^2},$$
$$I_{12}=I_{14}=I_{24}=I_{34}=0.$$

Therefore, $100(1-\alpha)\%$ approximately confidence interval for system reliability R based on MLE is as follows;

$$\left(\hat{R}_{MLE}-z_{\alpha/2}\cdot\sqrt{\hat{\Lambda}\cdot I(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3,\hat{\mu})\cdot\hat{\Lambda}'/n},\ \hat{R}_{MLE}+z_{\alpha/2}\cdot\sqrt{\hat{\Lambda}\cdot I(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3,\hat{\mu})\cdot\hat{\Lambda}'/n}\right). \tag{2.13}$$

3. Numerical Example

In this section, we present a numerical example by giving a data set which is generated by computer. We generate a strength sample of size 30 from BVP with parameter ($\lambda_1=1.0,\ \lambda_2=1.0,\ \lambda_3=0.5$) and stress sample from pareto with parameter $\mu=3.0$. The data is given Table 1 in the form of triplet (x_1, x_2, y).

MLEs of the parameters in BVP model are $\hat{\lambda}_1=0.9230$, $\hat{\lambda}_2=0.9911$, $\hat{\lambda}_3=0.4496$.

The estimate of variance-covariance matrix of $(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3)$ is

$$\begin{pmatrix} 0.0519 & & \\ 0 & 0.0568 & \\ -0.0089 & -0.0084 & 0.0273 \end{pmatrix}$$

The MLE of μ is $\hat{\mu}=2.8557$ and its estimate of variance is $Var(\hat{\mu})=0.2718$.

The estimate of system reliability is $\hat{R}_{MLE}=0.7929$ and its estimate of variance is $Var(\hat{R}_{MLE})=0.0023$. Hence, 95% confidence interval for system reliability is (0.6973, 0.8884).

In a similar way one can extend numerically the MLE of R when the common stress(Y) is other distribution or the strengths(X_1, X_2) are other distribution.

<Table 1> Generated samples (x_1, x_2, y) from BVP distribution and univariate pareto distribution.

i	x_{1i}	x_{2i}	y_i	i	x_{1i}	x_{2i}	y_i
1	1.1948	1.1948	1.1618	16	2.3743	4.2095	1.1971
2	1.4835	1.7787	1.1800	17	1.9545	1.7032	1.3059
3	1.2030	1.0944	1.2560	18	6.5830	1.6078	1.0236
4	1.7679	3.7424	1.1522	19	1.3265	3.7488	1.2186
5	2.1678	1.0869	1.5690	20	1.6034	5.9517	1.2444
6	1.0371	2.2421	1.0788	21	1.3295	2.2592	1.2926
7	1.2673	14.3983	2.3982	22	2.0184	2.0184	1.3496
8	3.4178	3.4178	1.2877	23	1.0788	1.0788	1.2043
9	6.7813	2.0274	3.6402	24	1.3653	1.3653	1.1484
10	2.4614	1.5689	1.3307	25	1.3557	2.7871	1.0717
11	7.0516	2.8727	4.0073	26	1.6987	3.4598	1.3992
12	3.2998	1.6695	1.7874	27	3.3873	1.3216	1.1103
13	1.4810	1.0360	3.6195	28	1.1978	1.1978	1.0505
14	3.2017	1.4355	1.3401	29	1.2106	1.2627	1.2697
15	1.5151	1.1113	1.0062	30	6.1369	1.1722	1.5972

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