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Distributivity of fuzzy numbers under t-norm based fuzzy arithmetic operations

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Abstract

Computation with fuzzy numbers is a prospective branch of a fuzzy set theory regarding the data processing applications. In this paper we consider an open problem about distributivity of fuzzy quantities based on the extension principle suggested by Mareš (1997). Indeed, we show that the distributivity on the class of fuzzy numbers holds and min-norm is the only continuous t-norm which holds the distributivity under t-norm based fuzzy arithmetic operations.

Key Words : Arithmetic operations; Fuzzy quantity

In this paper, we follow the same notations that in Mareš (1997).

1. Notations and basic notions

In the whole paper we denote by R the set of real numbers.

Any fuzzy subset a of R is called a *fuzzy quantity* with *membership function* $\mu_a : R \to [0,1]$ if and only if

$$\exists x_0 \in R : \mu_a(x_0) = 1,$$

$$\exists x_1, x_2 \in R,\ x_1 < x_2,\quad \forall x \notin [x_1, x_2] : \mu_a(x) = 0.$$

We denote by $\mathcal{R}$ the set of all fuzzy quantities.

A fuzzy number is a fuzzy quantity which is convex which means that an α

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-cut $a_\alpha = \{\mu_a \geq \alpha\} = [a_l^\alpha, a_r^\alpha]$ yields the property of nesting : that is

$$(\alpha' < \alpha) \rightarrow (a_l^{\alpha'} \leq a_l^\alpha,\ \ a_r^{\alpha'} \leq a_r^\alpha).$$

There are many different classes of fuzzy numbers.

Let $a \in R$ and let there exist real numbers $a_1 \leq a_0 \leq a'_0 \leq a_2 \in R$ such that

$$\mu_a(x) = \begin{cases} 0 & \text{for } x < a_1 \ \ or\ x > a_2, \\ 1 & \text{for } a_0 \leq x \leq a'_0, \\ \dfrac{x - a_1}{a_0 - a_1} & \text{for } x \in [a_1, a_0), \\ \dfrac{x - a_2}{a'_0 - a_2} & \text{for } x \in (a'_0, a_2]. \end{cases}$$

Then a is called *trapezoidal*. Equality $a_1 = a_0$ naturally means $\mu_a(x) = 0$ for $x < a_0$, $\mu_a(a_0) = 1$ and, analogously, if $a_2 = a'_0$ then $\mu_a(x) = 0$ for $x > a'_0$, $\mu_a(a'_0) = 1$. If $a_0 = a'_0$ then the fuzzy quantity a is called *triangular*.

The equality $a = b$ for $a, b \in \mathcal{R}$ means $\mu_a(x) = \mu_b(x)$ for all $x \in R$.

If $r \in R$ then we denote by $\langle r \rangle$ the degenerated fuzzy quantity defined by

$$\mu_{\langle r \rangle}(r) = 1,\ \ \mu_{\langle r \rangle}(x) = 0 \ \text{ for } x \neq r,\ x \in R.$$

If $a \in \mathcal{R}$ then $-a \in \mathcal{R}$ is the fuzzy quantity defined by

$$\mu_{-a}(x) = \mu_a(-x) \quad \text{for all } x \in R.$$

The elementary arithmetic operations over fuzzy quantities are derived from so called *extension principle*.

Definition 1 (*Extension Principle*). Let $f: R \times R \rightarrow R$ be a binary operation over real numbers. Then it can be extended to the operation over fuzzy quantities, $f: \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$. If we denote for $a, b \in \mathcal{R}$ the quantity $c = f(a, b)$ then the membership function μ_c is derived from the membership functions μ_a and μ_b by

$$\mu_c(z) = \sup[\min(\mu_a(x), \mu_b(y)) : x, y \in R,\ z = f(x, y)].$$

2. Arithmetic operations

The elementary binary arithmetic operations with fuzzy quantities or with fuzzy quantities or with crisp and fuzzy numbers are mostly based on the extension principle (cf. Dubois & Prade, 1988, Fullér, 1991).

Let $r \in R$, $a \in \mathcal{R}$ then $r + a \in \mathcal{R}$ is defined by

$$\mu_{r+a}(x) = \mu_a(x-r) \quad \text{for any} \quad x \in R.$$

If $a, b \in \mathcal{R}$ then $a \oplus b \in \mathcal{R}$ is defined by

$$\mu_{a \oplus b}(x) = \sup_{y \in R}(\min(\mu_a(y), \mu_b(x-y))), \quad x \in R.$$

It is easy to see that $r + a = \langle r \rangle \oplus a$.

Let $r \in R$, $a \in \mathcal{R}$ then $r \cdot a \in \mathcal{R}$ is defined for $x \in R$ by

$$\mu_{r \cdot a}(x) = \begin{cases} \mu_a(\frac{x}{r}) & \text{for} \quad r \neq 0 \\ \mu_{\langle 0 \rangle}(x) & \text{for} \quad r = 0. \end{cases}$$

If $a, b \in \mathcal{R}$ then $a \odot b \in \mathcal{R}$ is defined by

$$\mu_{a \odot b}(x) = \sup_{y \in R_0}(\min(\mu_a(y), \mu_b(\frac{x}{y}))),$$

$$x \in R_0 = R - \{0\},$$

$$\mu_{a \odot b}(0) = \max(\mu_a(0), \mu_b(0)).$$

It is easy to see that $r \cdot a = \langle r \rangle \odot a$ for $r \neq 0$.

Generally, all operations over fuzzy quantities given above generalize the operations over crisp operands. For degenerated fuzzy quantities, $a = \langle x \rangle$, $b = \langle y \rangle$, $x, y \in R$, $r \in R$,

$$r + a = \langle r + x \rangle, \qquad a \oplus b = \langle x + y \rangle,$$

$$r \cdot a = \langle r \cdot x \rangle, \qquad a \odot b = \langle x \cdot y \rangle.$$

3. Survey of properties

The main problem with processing fuzzy quantities are connected with the validity of group properties and distributivity. It is natural to consider $-a$ for the opposite to a but $a \oplus (-a)$ is not equal $\langle 0 \rangle$. More principle problem is connected with one of distributivity conditions. Namely, if $r_1, r_2 \in R$ and $a \in \mathcal{R}$ then $(r_1 + r_2) \cdot a$ is not generally equal to $(r_1 \cdot a) \oplus (r_2 \cdot a)$. It means that $a \oplus a$ need not be the same like $2 \cdot a$.

There is a method of avoiding there imperfectness:

We say that s is 0-symmetric if and only if

$$\mu_s(x) = \mu_s(-x) \quad \text{for all } x \in R.$$

The set of all 0-symmetric fuzzy quantities is denoted by S_0.

Let $a, b \in \mathcal{R}$. Then we say that a is additively equivalent to b, and write $a \sim_{\oplus} b$ if and only if there exist $s_1, s_2 \in S_0$. such that

$$a \oplus s_1 = b + s_2 .$$

Then it can be easily seen that for $a \in \mathcal{R}$, $s \in S_0$,

$$a \oplus s \sim_{\oplus} a, \quad s \sim_{\oplus} \langle 0 \rangle$$

$$a \oplus (-a) \sim_{\oplus} s, \quad s \odot a \sim_{\oplus} s$$

Namely, the 0-symmetric ones are able to play the role of fuzzy zero (see Kaufmann & Gupta, 1991).

Fuzzy quantity $b \in \mathcal{R}$ is call *almost trapezoidal* if and only if there exist trapezoial fuzzy quantity a such that $a \sim_{\oplus} b$.

If b is almost trapezoidal then for $r_1, r_2 \in \mathcal{R}$

$$(r_1 + r_2) \cdot b \sim_{\oplus} r_1 \cdot b \oplus r_2 \cdot b.$$

Here, an open problem suggested by Mareš (1997) is the validity of $(r_1 + r_2) \cdot a \sim_{\oplus} r_1 \cdot a \oplus r_2 \cdot a$ for more general fuzzy quantity a.

4. A proof of the open problem

In this section, we prove the following main result.

Theorem 1. Let a be a fuzzy number, then for $r_1, r_2 \in R$

$$(r_1 + r_2) \cdot a \sim_{\oplus} r_1 \cdot a \oplus r_2 \cdot a.$$

The following lemma is easy to check.

Lemma 1. Let $a, b \in \mathcal{R}$ then $a \oplus (-b) = s$ for $s \in S_0$ if and only if $a \sim_{\oplus} b$.

The following two results are well-known.

Resolution Theorem(Kaufmann & Gupta, 1991). Let $a \in \mathcal{R}$ and $a_\alpha = [a_1^\alpha, a_2^\alpha]$, then

$$a = \bigcup_\alpha \alpha \cdot a_\alpha \text{ where } \alpha \cdot a_\alpha = \alpha \wedge a_\alpha.$$

Theorem 2 (Nguyen, 1978). Let $f: R \times R \rightarrow R$ be a continuous function and let a and b be fuzzy numbers. Then

$$(f(a, b))_\alpha = f(a_\alpha, b_\alpha)$$

where $f(a_\alpha, b_\alpha) = \{f(x_1, x_2) \mid x_1 \in a_\alpha,\ x_2 \in b_\alpha\}$.

Let $f(x, y) = x + y$, $f(x, y) = xy$ and let $a_\alpha = [a_1^\alpha, a_2^\alpha]$ and $b_\alpha = [b_1^\alpha, b_2^\alpha]$ be two fuzzy numbers. Applying above theorem we get

$$\begin{aligned}
(a \oplus b)_\alpha &= a_\alpha + b_\alpha = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha], \\
(a \odot b)_\alpha &= a_\alpha b_\alpha = [\min\{a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha\}, \\
&\qquad \max\{a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha\}].
\end{aligned}$$

Proof of Theorem 1. By Lemma 1, it is sufficient to show

$$(r_1 + r_2) \cdot a \oplus ((-r_1) \cdot a \oplus (-r_2) \cdot a) = s \in S_0,$$

and hence, by Resolution Theorem, it is sufficient to show that for $\alpha \in [0,1]$, $((r_1+r_2)\cdot a \oplus ((-r_1)\cdot a \oplus (-r_2)\cdot a))_\alpha$ is symmetric interval.

Now noting that $(r\cdot a)_\alpha = [\min\{ra_1^\alpha, ra_2^\alpha\}, \max\{ra_1^\alpha, ra_2^\alpha\}]$, we have, by Lemma 2,

$$((r_1+r_2)\cdot a)_\alpha = [\min\{(r_1+r_2)a_1^\alpha, (r_1+r_2)a_2^\alpha\}, \max\{(r_1+r_2)a_1^\alpha, (r_1+r_2)a_2^\alpha\}]$$

and

$$\begin{aligned}
&((-r_1)\cdot a \oplus (-r_2)\cdot a)_\alpha \\
&= [\min\{(-r_1)a_1^\alpha + (-r_2)a_1^\alpha, (-r_1)a_2^\alpha + (-r_2)a_1^\alpha, \\
&\qquad (-r_1)a_1^\alpha + (-r_2)a_2^\alpha, (-r_1)a_2^\alpha + (-r_2)a_2^\alpha\}, \\
&\quad \max\{(-r_1)a_1^\alpha + (-r_2)a_1^\alpha, (-r_1)a_2^\alpha + (-r_2)a_1^\alpha, \\
&\qquad (-r_1)a_1^\alpha + (-r_2)a_2^\alpha, (-r_1)a_2^\alpha + (-r_2)a_2^\alpha\}],
\end{aligned}$$

and hence

$$\begin{aligned}
&((r_1+r_2)\cdot a \oplus ((-r_1)\cdot a \oplus (-r_2)\cdot a))_\alpha \\
&= [\min\{(r_1+r_2)(a_1^\alpha - a_2^\alpha), (r_1+r_2)(a_2^\alpha - a_1^\alpha), \\
&\qquad r_1(a_1^\alpha - a_2^\alpha), r_1(a_2^\alpha - a_1^\alpha)\}, \\
&\quad \max\{(r_1+r_2)(a_1^\alpha - a_2^\alpha), (r_1+r_2)(a_2^\alpha - a_1^\alpha), \\
&\qquad r_1(a_1^\alpha - a_2^\alpha), r_1(a_2^\alpha - a_1^\alpha)\}].
\end{aligned}$$

Indeed,

$$\begin{aligned}
&((r_1+r_2)\cdot a \oplus ((-r_1)\cdot a \oplus (-r_{2)}\cdot a))_\alpha \\
= \ & [-(r_1+r_2)(a_2^\alpha - a_1^\alpha), (r_1+r_2)(a_2^\alpha - a_1^\alpha)] \quad \text{for } r_1, r_2 \geq 0, \\
& [(r_1+r_2)(a_2^\alpha - a_1^\alpha), -(r_1+r_2)(a_2^\alpha - a_1^\alpha)] \quad \text{for } r_1, r_2 \leq 0, \\
& [-r_2(a_2^\alpha - a_1^\alpha), r_2(a_2^\alpha - a_1^\alpha)] \quad \text{for } r_1 \leq 0 \leq r_2 \text{ and } |r_1| \leq |r_2|, \\
& [r_1(a_2^\alpha - a_1^\alpha), -r_1(a_2^\alpha - a_1^\alpha)] \quad \text{for } r_1 \leq 0 \leq r_2 \text{ and } |r_1| \geq |r_2|, \\
& [r_2(a_2^\alpha - a_1^\alpha), -r_2(a_2^\alpha - a_1^\alpha)] \quad \text{for } r_2 \leq 0 \leq r_1 \text{ and } |r_1| \leq |r_2|, \\
& [-r_1(a_2^\alpha - a_1^\alpha), r_1(a_2^\alpha - a_1^\alpha)] \quad \text{for } r_2 \leq 0 \leq r_1 \text{ and } |r_1| \geq |r_2|
\end{aligned}$$

is symmetric interval which completes the proof.

5. Distributivity of fuzzy numbers under t-norm based fuzzy arithmetic operations

It is also needed to consider the distributivity of fuzzy numbers under t-norm based fuzzy arithmetic operations.

Definition 2. A mapping $T:[0,1]\times[0,1]\to[0,1]$ is a triangular norm (t-norm for short) if and only if it is symmetric, associative, non-decreasing in each argument and $T(a,1)=a$ for all $a\in[0,1]$.

In the definition of extension principle one can use any t-norm for modeling the conjunction operator.

Definition 1′. Let T be a t-norm and let $f: R\times R\to R$ be a binary operation over real numbers. Then it can be extended to the operation over fuzzy quantities, $f: \mathcal{R}\times\mathcal{R}\to\mathcal{R}$. If we denote for $a,b\in\mathcal{R}$ the quantity $c=f(a,b)$ then the membership function μ_c is derived from the membership functions μ_a and μ_b by

$$\mu_c(z)=\sup[\,T(\mu_a(x),\mu_b(y)) : x,y\in R,\ z=f(x,y)].$$

Specially, if T is a t-norm and $f(x_1,x_2)=x_1+x_2$ is the addition operation on the real line then the sup-t extended sum of a and b, called T-sum and denoted by $a\oplus_T b$, is defined by

$$\mu_{a\oplus_T b}(x)=\sup_{y\in R}(\,T(\mu_a(y),\mu_b(x-y))),\quad x\in R,$$

and if $f(x_1,x_2)=x_1x_2$ is the multiplication operation on the real line the T-product of a and b, denoted by $a\odot_T b$, is defined by

$$\mu_{a\odot_T b}(x)=\sup_{y\in R_0}(\,T(\mu_a(y),\mu_b(x/y))),\quad x\in R.$$

In this section, we show that, for every continuous t-norm, the distributivity of fuzzy numbers under t-norm based fuzzy arithmetic operations dose not hold.

We need the following two known results.

Lemma 3 **(Fullér, 1998).** $T(x,x)=x$ holds for any $x\in[0,1]$ if and only if T is the minimum norm.

The following theorem illustrates that if we use an arbitrary t-norm instead of min-norm in Zadeh's extension principle then we obtain result similar to those of Nguyen(1978).

Theorem 3 (Fullér & Keresatfalvi, 1991). If $f: R\times R\rightarrow R$ is continuous and t-norm T is upper semicontinuous, then

$$(f(a,b))_{\alpha}=\bigcup_{T(\xi,\eta)\geq\alpha}f(a_{\xi},b_{\eta}),\ \ \alpha\in(0,1]$$

hold for $a,b\in\mathcal{R}$.

Now, let $\mu_a(x)=1-|x|$ on $[-1,1]$ and zero otherwise and let T be continuous t-norm which is not minimum norm. Then, by Lemma 3, there exists $x_0\in(0,1)$ such that $T(x_0,x_0)<x_0$, and hence by continuity of T, for some $\varepsilon>0$.

$$\inf\{\max(\xi,\eta)|\,T(\xi,\eta)\geq x_0\}=x_0+\varepsilon.$$

We also have that, since minimum norm is the biggest t-norm,

$$\inf\{\min(\xi,\eta)|\,T(\xi,\eta)\geq x_0\}=x_0.$$

Let $r_1=r_2=1$, then $((r_1+r_2)\cdot a)_{x_0}=[2(x_0-1),2(1-x_0)]$ and, $(r_1\cdot a)_{\alpha}=(r_2\cdot a)_{\alpha}=[\alpha-1,1-\alpha]$ and hence, by Theorem 3,

$$(r_1\cdot a\oplus_T r_2\cdot a)\subset[2(x_0-1)+\varepsilon,\ 2(1-x_0)-\varepsilon].$$

Therefore $(r_1+r_2)\cdot a\neq r_1\cdot a\oplus_T r_2\cdot a$.

6. Conclusion

The fuzzy quantities do not form an additive group if the strict equality is demanded and $\langle 0\rangle$ is considered for the zero. In this paper, we have shown that the set of fuzzy numbers become linear spaces by substituting strict equality by additive equivalence and taking the class S_0 for "*fuzzy zero*" and min-norm is the only continuous t-norm which holds the distributivity under t-norm based fuzzy arithmetic operation on the class of fuzzy numbers. We note that most of theory and applications can be applied within the set of fuzzy numbers without any difficulties. So this generalization makes sense.

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