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A convergence of fuzzy random variables[1)]

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Abstract

In this paper, a general convergence theorem of fuzzy random variables is considered. Using this result, we can easily prove the recent result of Joo et al. (2001) and generalize the recent result of Kim(2000).

Key words : Fuzzy number; Fuzzy random variable; Strong law of large numbers

1. Introduction

In recent years, strong laws of large numbers for sums of fuzzy random variables have received much attention by several people. A SLLN for sums of i.i.d. fuzzy random variables was obtained by Kruse (1982), and a SLLN for sums of independent fuzzy random variables was obtained by Miyakoshi and Shimbo (1984), Klement, Puri and Ralescu (1986). Also, Inoue (1991) obtained a SLLN for sums of independent tight fuzzy random sets, and Hong and Kim (1994) proved Marcinkiewicz-type law of large numbers. Many other papers are related with this topic. Recently, Joo, Lee and Yoo (2001) generalized a strong law of large numbers for sums of stationary and ergodic processes to the case of fuzzy random variables and Kim (2000) obtained a strong law of large numbers for sums of levelwise independent and levelwise identically distributed fuzzy random variables.

In this paper, we consider a general convergence theorem of fuzzy random variables, Using this result, we can easily prove the result of Joo et al. (2001) and generalize the result of Kim (2000). Section 2 is devoted to describe some basic concepts of fuzzy random variables. Main results are given in Section 3.

2. Preliminaries

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Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u}: R \to [0,1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$

(2) $\tilde{u}$ is upper semicontinuous.

(3) $\text{supp}\,\tilde{u} = cl\{x \in R \mid \tilde{u}(x) > 0\}$ is compact.

(4) $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$

Let $F(R)$ be the family of all fuzzy numbers. For a fuzzy set $\tilde{u}$, if we define

$$L_\alpha \tilde{u} = \begin{cases} \{x \mid \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ \text{supp}\,\tilde{u}, & \alpha = 0, \end{cases}$$

then, it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1 \tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy number, a fuzzy number $\tilde{u}$ is completely determined by the end points of the intervals $L_\alpha \tilde{u} = [u_\alpha^1, u_\alpha^2]$.

The following theorem(see Goetschel and Voxman, 1986) implies that we can identify a fuzzy number $\tilde{u}$ with the parameterized representation

$$\{(u_\alpha^1, u_\alpha^2) \mid 0 \leq \alpha \leq 1\}.$$

Theorem 2.1. For $\tilde{u} \in F(R)$, denote $u^1(\alpha) = u_\alpha^1$ and $u^2(\alpha) = u_\alpha^2$ by considering as functions of $\alpha \in [0,1]$. Then

(1) u^1 is a bounded increasing function on [0,1].

(2) u^2 is a bounded increasing function on [0,1].

(3) $u^1(1) \leq u^2(1)$.

(4) u^1 and u^2 are left continuous on $[0,1]$ and right continuous at 0.

(5) If v^1 and v^2 satisfy above (1)-(4), then there exists a unique $\tilde{v} \in F(R)$ such that $v_\alpha^1 = v^1(\alpha), v_\alpha^2 = v^2(\alpha)$.

The addition and scalar multiplication on $F(R)$ are defined as usual;

$$(\tilde{u}+\tilde{v})(z)=\sup_{x+y=z}\min(\tilde{u}(x),\tilde{v}(x)),$$

$$(\lambda\tilde{u})(z)=\begin{cases}\tilde{u}(z/\lambda), & \lambda\neq 0,\\ \tilde{0}, & \lambda=0,\end{cases}$$

for $\tilde{u},\tilde{v}\in F(R)$ and $\lambda\in R$, where $\tilde{0}=I_{\{0\}}$ is the indicator function of $\{0\}$. It follows that if $\tilde{u}=\{(u^{1_\alpha},u^{2_\alpha})\mid 0\le\alpha\le 1\}$ and $\tilde{v}=\{(v^{1_\alpha},v^{2_\alpha})\mid 0\le\alpha\le 1\}$, then

$$\tilde{u}+\tilde{v}=\{(u^{1_\alpha}+v^{1_\alpha},u^{2_\alpha}+v^{2_\alpha})\mid 0\le\alpha\le 1\}$$

$$\lambda\tilde{u}=\{(\lambda u^{1_\alpha},\lambda u^{2_\alpha})\mid 0\le\alpha\le 1\}\text{ for }\lambda\ge 0.$$

Now, we define the metric d_∞ on $F(R)$ by

$$d_\infty(\tilde{u},\tilde{v})=\sup_{0\le\alpha\le 1}h(L_\alpha\tilde{u},L_\alpha\tilde{v}),$$

where h is Hausdorff metric defined as

$$h(L_\alpha\tilde{u},L_\alpha\tilde{v})=\max(|u^1_\alpha-v^1_\alpha|,|u^2_\alpha-v^2_\alpha|).$$

The norm of $\tilde{u}\in F(R)$ is defined by

$$\|\tilde{u}\|=d_\infty(\tilde{u},\tilde{0})=\max(|u^1_0|,\ |u^2_0|).$$

Then it is well-known that $F(R)$ is complete but nonseparable with respect to the metric d_∞. Joo and Kim (2000) introduced a metric d_s in $F(R)$ which makes it a separable metric space as follows.

Definition 2.1. Let T denote the class of strictly increasing, continuous mapping of [0, 1] onto itselt. For $\tilde{u},\tilde{v}\in F(R)$, we define

$$d_s(\tilde{u},\tilde{v})=\inf\{\varepsilon:\text{ there exists a } t \text{ in} T \text{ such that } \sup_{0\le\alpha\le 1}|t(\alpha)-\alpha|\le\varepsilon\text{ and } d_\infty(\tilde{u},t\circ\tilde{v})\le\varepsilon\},$$

where $t\circ\tilde{v}$ denotes the composition of $\tilde{v}$ and t.

3. Main results

Throughout this section, we assume that the space $F(R)$ is considered as the metric space endowed with the metric d_s, unless otherwise stated. Also, we denote by $\mathcal{B}_s$ the Borel σ-field of $F(R)$ generated by the metric d_s.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A fuzzy number valued function $\mathcal{X}: \Omega \to F(R)$ is called a fuzzy random variable if it is measurable, i.e.,

$$\mathcal{X}^{-1}(B) = \{\omega : \mathcal{X}(\omega) \in B\} \in \mathcal{A} \text{ for every } B \in \mathcal{B}_s.$$

If we denote $\mathcal{X}(\omega) = \{(X_\alpha^1(\omega), X_\alpha^2(\omega)) \mid 0 \le \alpha \le 1\}$, then it is known that $\mathcal{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0,1]$, X_α^1 and X_α^2 are random variables in the usual sense. A fuzzy random variable $\mathcal{X} = \{(X_\alpha^1, X_\alpha^2) \mid 0 \le \alpha \le 1\}$ is called integrable if for each $\alpha \in [0,1]$, X_α^1 and X_α^2 are integrable, equivalently, $\int \|\mathcal{X}\| dP < \infty$. In this case, the expectation of $\mathcal{X}$ is the fuzzy number $E\mathcal{X}$ defined by

$$E\mathcal{X} = \{(EX_\alpha^1, EX_\alpha^2) \mid 0 \le \alpha \le 1\}$$

Theorem 3.1. Let $\{\mathcal{X}_n\} = \{(X^1_{n\alpha}, X^2_{n\alpha}) \mid 0 \le \alpha \le 1\}$ be a sequence of fuzzy random variables and $\tilde{u} = \{(u_\alpha^1, u_\alpha^2) \mid 0 \le \alpha \le 1\}$ be a fuzzy number with $\|\tilde{u}\| < \infty$. Suppose that

(1) $X^1_{n\alpha} \to u^1_\alpha$ a.s and $X^2_{n\alpha} \to u^2_\alpha$ a.s for any $\alpha \in [0,1]$

(2) $X^1_{n\alpha^+} \to u^1_{\alpha^+}$ a.s and $X^2_{n\alpha^-} \to u^2_{\alpha^-}$ a.s for every discontinuity point of u_1^α and u_2^α, respectively.

Then we have

$$\lim_{n\to\infty} d_\infty(\mathcal{X}_n, \tilde{u}) = 0 \text{ a.s.}$$

We need the following lemma given in Joo et al. (2001).

Lemma 3.1. Let $u = \{(u_\alpha^1, u_\alpha^2) \mid 0 \le \alpha \le 1\}$ with $\|u\| < \infty$ and $\varepsilon > 0$ be given.

(1) Then there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of $[0,1]$ such that

$u^1_{\alpha_i} - u^1_{\alpha^+_{i-1}} \leq \varepsilon$ for all $i = 1, 2, \cdots, r.$

(2) Similar statements hold for u^2_α.

Proof of Theorem 3.1. Let $\varepsilon > 0$ be arbitrary fixed. By Lemma 3.1, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of $[0, 1]$ such that $u^1_{\alpha_i} - u^1_{\alpha^+_{i-1}} \leq \varepsilon$ for all $i = 1, 2, \cdots, r$. Let $A_k = \{X^1_{n\alpha_k} \to u^1_{\alpha_k}$ and $X^1_{n\alpha^+} \to u^1_{\alpha^+}$ for all discontinuity points of $u^1_\alpha\}$ and $A_\varepsilon = \bigcap_{k=1}^{r} A_k$, then by the assumption $P(A_k) = 1,\ k = 1, 2, \cdots, r,$ and hence $P(A_\varepsilon) = 1$. Then for any given $w \in A_\varepsilon$, there exists $N(w)$ such that for $n \geq N(w)$,

$$\sup_{k=1,2,\cdots,r} \{|X^1_{n\alpha_k}(w) - u^1_{\alpha_k}|,\ |X^1_{n\alpha^+_k}(w) - u^1_{\alpha^+_k}|\} \leq \varepsilon.$$

Now, let $\alpha \in (\alpha_{k-1}, \alpha_k]$, then for $n \geq N(w)$,

$$\begin{aligned} X^1_{n\alpha}(w) - u^1_\alpha &\leq X^1_{n\alpha_k}(w) - u^1_{\alpha^+_{k-1}} \\ &\leq u^1_{\alpha_k} + \varepsilon - u^1_{\alpha^+_{k-1}} \leq 2\varepsilon \end{aligned}$$

and

$$\begin{aligned} u^1_\alpha - X^1_{n\alpha}(w) &\leq u^1_{\alpha_k} - X^1_{n\alpha^+_{k-1}}(w) \\ &\leq u^1_{\alpha_k} - (u^1_{\alpha^+_{k-1}} - \varepsilon) \leq 2\varepsilon. \end{aligned}$$

Hence

$$\sup_{\alpha \in (\alpha_{k-1}, \alpha_k]} |X^1_{n\alpha}(w) - u^1_\alpha| \leq 2\varepsilon.$$

Since k is arbitrary, we have

$$\sup_{\alpha \in [0,1]} |X^1_{n\alpha}(w) - u^1_\alpha| \leq 2\varepsilon.$$

Let $A = \bigcap_{n=1}^{\infty} A_{\frac{1}{n}}$, then $P(A) = 1$ and for any $w \in A$

$$\lim_{n\to\infty} \sup_{0\le\alpha\le 1} |X^1_{n\alpha}(w) - u^1_\alpha| = 0.$$

Similarly, it can be proved that

$$\lim_{n\to\infty} \sup_{0\le\alpha\le 1} |X^2_{n\alpha} - u^2_\alpha| = 0, \text{ a.s.}$$

which completes the proof.

Recently, Kim (2000) proved a SLLN for sums of levelwise independent and identically distributed fuzzy random variables. But his result is a special case of Theorem 1. If $\tilde{X}_n$ is a sequence of levelwise independent and levelwise identically distributed random variables with $E\|\tilde{X}_1\| < \infty$, then, it is easy to check that both $\{X^1_{n\alpha+}\}$ and $\{X^2_{n\alpha-}\}$ for $\alpha\in[0,1]$ are independent and identically distributed random variables, respectively, with $E|\tilde{X}^1{}_{n\alpha+}| < \infty$ and $E|\tilde{X}^2{}_{n\alpha-}| < \infty$. And it is also easy to check that for any $\alpha\in[0,1]$

$$\frac{1}{n}\sum_{i=1}^n X^1_{i\alpha+} \to EX^1_{\alpha+} \text{ a.s.}$$

and

$$\frac{1}{n}\sum_{i=1}^n X^2_{i\alpha-} \to EX^2_{\alpha-} \text{ a.s.}$$

by Kolmogorov's strong law of large numbers and monotone convergence Theorem. It is also noted that the set of discontinuity point of EX^1_α and EX^2_α is at most countable. Now, using Theorem 1 we have the following generalized result of Kim (2000) as a corollary.

Corollary 3.1. Let $\{\tilde{X}_n\}$ be a sequence of levelwise independent and levelwise identically distributed fuzzy random variables, with $E\|\tilde{X}_1\| < \infty$. Then we have

$$d_\infty\left(\frac{1}{n}\sum_{i=1}^n \tilde{X}_i, E\tilde{X}_1\right) \to 0 \text{ a.s.}$$

Remark. The condition that $EX^1_{1\alpha}$ and $EX^2_{1\alpha}$ are continuous as functions of α in Kim's result is not needed.

Recently Joo et al. (2001) proved a SLLN for sums of stationary and ergodic fuzzy random variables. With similar arguments as above, noting that for each $\alpha \in [0,1]$, $\{X^1_{n\alpha}\}$, $\{X^1_{n\alpha+}\}$, $\{X^2_{n\alpha}\}$ and $\{X^2_{n\alpha-}\}$ are sequences of stationary and ergodic random variables under the assumption that $\{\tilde{X}_n\}$ is a sequence of stationary and ergodic fuzzy random variables, we also have Joo's result as a corollary by Theorem 1.

Corollary 3.2. Let X_n be a sequence of stationary fuzzy random variables. If $\{\tilde{X}_n\}$ is ergodic and $E\|\tilde{X}_1\| < \infty$, then

$$d_\infty\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i, E\tilde{X}_1\right) \to 0 \text{ a.s.}$$

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