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Large Sample Tests for Independence and Symmetry in the Bivariate Weibull Model under Random Censorship

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Abstract

In this paper, we consider two components system which the lifetimes have a bivariate weibull distribution with random censored data. Here the censoring time is independent of the lifetimes of the components. We construct large sample tests for independence and symmetry between two-components based on maximum likelihood estimators and the natural estimators. Also we present a numerical study.

Key Words : Bivariate weibull distribution; Random censorship.

1. Introduction

In many studies for the reliability of two components system, occasionally, independence assumption is not applicable in the practical situation. Naturally, it is more realistic to assume some forms of dependence among the components of the system. This dependence among the components arise from common environmental shocks and stress, or from components depending on common sources of power, and so on. (See Esary and Proschan (1970)).

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As the forms of dependence among the components in two-components system, bivariate weibull distribution(BVW) is a versatile family of life distributions in view of its physical interpretation and its flexibility for empirical fit, and has been extensively applied to analysis of life data concerning many types of manufactured items. The examples of a bivariate weibull(BVW) distribution can be visualized in many contexts, such as the times to first and second failures of a repairable device, the breakdown times of dual generators in a power plant, or the survival times of the organs in a two-organ system, such as lungs or kidneys, in the human body.

Lu and Bhattacharyya (1988, 1990) initially introduced some new construction of BVW distributions. Lu(1989) derived BVW distribution as extensions of the Freund and Marshall-Olkin bivariate exponential models. Cho, Cha and Lee(2003) obtained the system reliability from stress-strength relationship.

Above all authors studied the case of complete data. Now we assume that there occur censored observations in lifetime. For example, there is a system which can not measure some characteristics of the lifetime of an item above a certain value. In this case, real lifetime may exceed the measured value, but we do not know the exact lifetime.

In this paper, we assume that the lifetimes of components in two components system follow a BVW distribution with censored data. Here, the censoring time is independent of the lifetimes of the components. And we construct a large sample tests for independence and symmetry between two components based on maximum likelihood estimators and natural estimators, respectively. Also we present a numerical example by giving a data set which is generated by computer.

2. Preliminaries

Let random variable (X, Y) be lifetimes of two components that follow a BVW distribution with parameter $(\xi_1, \xi_2, \xi_3, \psi)$. Then the joint probability density function of (X, Y) is given as

$$f(x, y; \xi_1, \xi_2, \xi_3, \psi) = \begin{cases} \xi_1(\xi_2+\xi_3)\psi^2 x^{\psi-1} y^{\psi-1} \exp[-\xi_1 x^{\psi} - (\xi_2+\xi_3) y^{\psi}], & 0 < x < y < \infty, \\ \xi_2(\xi_1+\xi_3)\psi^2 x^{\psi-1} y^{\psi-1} \exp[-(\xi_1+\xi_3) x^{\psi} - \xi_2 y^{\psi}], & 0 < y < x < \infty, \\ \xi_3 \psi x^{\psi-1} \exp[-\xi x^{\psi}], & 0 < x = y < \infty, \end{cases} \quad (1)$$

where $\xi_1, \xi_2, \xi_3, \psi > 0$ and $\xi = \xi_1 + \xi_2 + \xi_3$.

And the joint survival function of (X, Y) is given by

$$\overline{F}(x, y) = P(X > x, Y > y) = \exp[-(\xi_1 x^{\psi} + \xi_2 y^{\psi} + \xi_3 \max(x, y)^{\psi})]. \quad (2)$$

The above BVW distribution is not absolutely continuous with respect to Lebesgue measure on R^2. That is, there is provision for simultaneous failure of the both components, $P[X=Y]=\xi_3/\xi$. Also the marginal distribution of X is given by

$$\overline{F}(x)=P(X>x)=\exp[-(\xi_1+\xi_3)x^{\psi}], \tag{3}$$

which is the survival function of weibull with parameters $(\xi_1+\xi_3,\psi)$. By similar method, the marginal distribution of Y is given as $\overline{F}(y)=P(Y>y)$ $=\exp[-(\xi_2+\xi_3)y^{\psi}]$ which is the survival function of weibull with parameters $(\xi_2+\xi_3,\psi)$.

From (1)-(3), random variables X and Y are independent if and only if $\xi_3=0$. And X and Y are symmetric distributed if and only if $\xi_1=\xi_2$. Also BVW leads to the Marshall Olkin's bivariate exponential distribution if and only if $\psi=1$.

Suppose that there are n two component systems under study and ith pair of the components have life time (x_i,y_i) and a censoring time t_i. We assume that the censoring time t_i has a weibull distribution with parameter (ζ,ψ) which is independent of $f(x_i,y_i)$. Then ith observed lifetime (x_i,y_i) is given by

$$(x_i,y_i) = \begin{cases} (x_i,y_i), & \text{if } \max(x_i,y_i)<t_i \\ (x_i,t_i), & \text{if } x_i<t_i<y_i \\ (t_i,y_i), & \text{if } y_i<t_i<x_i \\ (t_i,t_i), & \text{if } t_i<\min(x_i,y_i). \end{cases} \tag{4}$$

Let $I(\cdot)$ be an indicator function. And we define n_j $(j=1,\cdots,6)$ as follows;

$$n_1 = \sum_{i=1}^{n} I(x_i<y_i<t_i),\ n_2 = \sum_{i=1}^{n} I(y_i<x_i<t_i),\ n_3 = \sum_{i=1}^{n} I(x_i=y_i<t_i),$$

$$n_4 = \sum_{i=1}^{n} I(x_i<t_i<y_i),\ n_5 = \sum_{i=1}^{n} I(y_i<t_i<x_i),\ n_6 = \sum_{i=1}^{n} I(\min(x_i,y_i)>t_i).$$

And let S_i, $i=1,\cdots,6$ be the set of elements as follows;

$$S_1=\{i\mid x_i<y_i<t_i,\ i=1,\cdots,6\},\ S_2=\{i\mid y_i<x_i<t_i,\ i=1,\cdots,6\},$$
$$S_3=\{i\mid x_i=x_i<t_i,\ i=1,\cdots,6\},\ S_4=\{i\mid x_i<t_i<y_i,\ i=1,\cdots,6\},$$
$$S_5=\{i\mid y_i<t_i<x_i,\ i=1,\cdots,6\},\ S_6=\{i\mid \min(x_i,y_i)>t_i,\ i=1,\cdots,6\}.$$

After some calculations, the expected value of n_j $(j=1,\cdots,6)$ can be obtained as follows;

$$E(n_1)=n\left(\frac{\xi_1}{\xi}+\frac{\zeta(\xi_2+\xi_3)}{\xi(\xi+\zeta)}-\frac{\zeta}{(\xi_2+\xi_3+\zeta)}\right),$$

$$E(n_2)=n\left(\frac{\xi_2}{\xi}+\frac{\zeta(\xi_1+\xi_3)}{\xi(\xi+\zeta)}-\frac{\zeta}{(\xi_1+\xi_3+\zeta)}\right),\ E(n_3)=\frac{n\xi_3}{\xi+\zeta},$$

$$E(n_4)=n\xi\left(\frac{1}{\zeta+\xi_2+\xi_3}-\frac{1}{\xi+\zeta}\right),\ E(n_5)=n\xi\left(\frac{1}{\zeta+\xi_1+\xi_3}-\frac{1}{\xi+\zeta}\right),$$

$$E(n_6)=\frac{n\zeta}{\xi+\zeta}.$$

Let $m_1=n_1+n_4$, $m_2=n_2+n_5$, $m_3=n_3$ and $m_4=n_6$. Then (m_1,m_2,m_3,m_4) is multinomial distribution with parameter $\left(n,\frac{\xi_1}{\xi+\zeta},\frac{\xi_2}{\xi+\zeta},\frac{\xi_3}{\xi+\zeta},\frac{\zeta}{\xi+\zeta}\right)$.

Then the likelihood function of the sample of size n is given by

$$\begin{aligned}L=&\ \xi_1^{n_1+n_4}\cdot\xi_2^{n_2+n_5}\cdot\xi_3^{n_3}\cdot(\xi_1+\xi_3)^{n_2}\cdot(\xi_2+\xi_3)^{n_1}\\&\cdot\phi^{2(n_1+n_2+n_4+n_5)+n_3+n_6}\cdot\zeta^{n_4+n_5+n_6}\cdot\Big[\prod_{i\in\{S_1\cup S_2\cup S_3\cup S_4\}}x_i\Big]^{(\phi-1)}\\&\cdot\Big[\prod_{i\in\{S_1\cup S_2\cup S_3\cup S_5\}}y_i\Big]^{(\phi-1)}\cdot\Big[\prod_{i\in\{S_3\}}(x_i=y_i)\Big]^{-(\phi-1)}\\&\cdot\exp\left[-\xi_1\sum_{i=1}^n x_i^\phi-\xi_2\sum_{i=1}^n yi^\phi-(\xi_3+\zeta)\sum_{i=1}^n\max(x_i,y_i)^\phi\right].\end{aligned}\tag{5}$$

In this paper, we focus only on BVW with fixed ϕ. To obtain the MLE's of $(\xi_1,\xi_2,\xi_3,\zeta)$, the likelihood equations are given by

$$\frac{n_1+n_4}{\xi_1}+\frac{n_2}{\xi_1+\xi_3}-\sum_{i=1}^n x_i^\phi=0,\tag{6}$$

$$\frac{n_2+n_5}{\xi_2}+\frac{n_1}{\xi_2+\xi_3}-\sum_{i=1}^n y_i^\phi=0,\tag{7}$$

$$\frac{n_3}{\xi_3}+\frac{n_2}{\xi_1+\xi_3}+\frac{n_1}{\xi_2+\xi_3}-\sum_{i=1}^n\max(x_i,y_i)^\phi=0.\tag{8}$$

$$\frac{n_4+n_5+n_6}{\zeta}-\sum_{i=1}^n\max(x_i,y_i)^\phi=0.\tag{9}$$

The likelihood equations (6)-(8) are not easy to solve. But we can obtain MLE's ($\hat{\xi}_1$, $\hat{\xi}_2$, $\hat{\xi}_3$) by either Newton-Raphson procedure or Fisher's method of scoring.

The Fisher information matrix is given by

$$I(\xi_1, \xi_2, \xi_3) = E\left[\frac{\partial^2 logL}{\partial\xi_i \partial\xi_j}\right] = n((I_{ij})) \ ; \ i, \ j=1,2,3, \tag{10}$$

where $I_{11} = \left(\frac{E(n_1)+E(n_4)}{n\xi_1^2} + \frac{E(n_2)}{n(\xi_1+\xi_3)^2}\right)$, $I_{12}=0$, $I_{13} = \frac{E(n_2)}{n(\xi_1+\xi_3)^2}$,

$$I_{22} = \left(\frac{E(n_2)+E(n_5)}{n\xi_2^2} + \frac{E(n_1)}{n(\xi_2+\xi_3)^2}\right), \quad I_{23} = \frac{E(n_1)}{n(\xi_2+\xi_3)^2},$$

$$I_{33} = \left(\frac{E(n_1)}{n(\xi_2+\xi_3)^2} + \frac{E(n_2)}{n(\xi_1+\xi_3)^2} + \frac{E(n_3)}{n\xi_3^2}\right).$$

Thus $\sqrt{n}(\hat{\underset{\sim}{\xi}} - \underset{\sim}{\xi})$ has asymptotic trivariate normal distribution with mean vector zero and covariance matrix $I^{-1}(\underset{\sim}{\xi}) = \frac{1}{n}((I^{ij}))$; $i,j=1,2,3$. Here, $\hat{\underset{\sim}{\xi}} = (\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3)$ and $\underset{\sim}{\xi} = (\xi_1, \xi_2, \xi_3)$.

3. Large Sample Tests for Independence and Symmetry

In this section, we construct two tests of hypothesis for independence and symmetry between two components based on MLE's and natural estimator (m_1, m_2, m_3, m_4).

3.1 Large Sample Test for Independence

Note that when $\xi_3 = 0$ we will have only one sided alternative $H_1: \xi_3 > 0$ and the test is equivalent to test for independence of X and Y. By section 2, $\hat{\xi}_3$ has asymptotic normal distribution with mean ξ_3 and variance I^{33}/n but I^{33} depends on the unknown parameters (ξ_1, ξ_2, ξ_3). We estimate it from the MLE's of (ξ_1, ξ_2, ξ_3) and construct the test statistic $\phi_{MLE}(X, Y) = \sqrt{n} \cdot \hat{\xi}_3/\sqrt{\hat{I}^{33}}$ which has asymptotic normal distribution. For $H_1: \xi_3 > 0$, we reject H_o with significance level α if

$$\phi_{MLE}(X, Y) = \sqrt{n} \cdot \hat{\xi}_3/\sqrt{\hat{I}^{33}} \ > \ z_{1-\alpha}, \tag{11}$$

where z_α is the $100 \cdot \alpha$ percentile of standard normal distribution.

We can also obtain a large sample test for independence, $H_o: \xi_3 = 0$ based on m_3 which is binomial($n, \xi_3/\xi$) and use the studentized test statistic as follows;

$$\phi_{M_i}(X, Y) = \sqrt{n} m_3/\sqrt{m_3(n-m_3)}, \tag{12}$$

which is asymptotic normal distribution. Hence for $H_1: \xi_3 > 0$, we reject H_o with significance level α if

$$\phi_{M_i}(X, Y) = \sqrt{n}m_3/\sqrt{m_3(n-m_3)} > z_{1-\alpha}. \quad (13)$$

3.2 Large Sample Test of Symmetry

Note that test for $H_o: \xi_1 = \xi_2$ is equivalent to test of symmetry of X and Y. We construct the tests of hypothesis for $H_o: \xi_1 = \xi_2$ based on $\hat{\xi}_1 - \hat{\xi}_2$ and $m_1 - m_2$. The exact distribution of these statistics are difficult to obtain but their asymptotic normal distributions can be obtained using the results of section 2.

We first consider the test based on $\hat{\xi}_1 - \hat{\xi}_2$. Now $\hat{\xi}_1 - \hat{\xi}_2$ has asymptotic normal distribution with mean $\xi_1 - \xi_2$ and variance $I^{11} + I^{22} - 2I^{12}$, but as $I^{11} + I^{22} - 2I^{12}$ depends on unknown parameters (ξ_1, ξ_2, ξ_3) we studentize and construct the test statistic $\pi_{MLE}(X, Y) = \sqrt{n}(\hat{\xi}_1 - \hat{\xi}_2)/\sqrt{\hat{I}^{11} + \hat{I}^{22} - 2\hat{I}^{12}}$ which is asymptotic normal distributed. For the alternative $H_1: \xi_1 \neq \xi_2$, we reject H_o with significance level α if

$$\pi_{MLE}(X, Y) = \left| \frac{\sqrt{n}(\hat{\xi}_1 - \hat{\xi}_2)}{\sqrt{\hat{I}^{11} + \hat{I}^{22} - 2\hat{I}^{12}}} \right| > z_{\alpha/2}. \quad (14)$$

In exactly similar way, if we use the studentized test statistic based on $m_1 - m_2$ which is asymptotic normal distribution, for the alternative $H_1: \xi_1 \neq \xi_2$, we reject H_o with significance level α if

$$\pi_{M_i}(X, Y) = \left| \sqrt{n}(n_1 - n_2)/\sqrt{4m_1m_2 + (m_3 + m_4)(m_1 + m_2)} \right| > z_{\alpha/2}. \quad (15)$$

4. Numerical Example

In this section, we present a numerical example by giving a data set which is generated by computer. We generate a random samples of size 30 from BVW with parameter ($\xi_1 = 1.0$, $\xi_2 = 1.2$, $\xi_3 = 0.2$, $\zeta = 2.0$). Also we generate random censored data of size 30 corresponding the lifetimes from weibull with parameters $\zeta = 2.0$, $\phi = 0.2$. Then the generated data is not symmetric and not independent between X and Y. That is, true hypotheses are $H_1: \xi_3 > 0$ for independence and $H_1: \xi_1 \neq \xi_2$ for symmetry. The data is given Table 1. In Table 1, * indicates censored data.

<Table 1> Generated sample from BVW

i	x_i	y_i	i	x_i	y_i
1	0.5671	0.0607*	16	1.1776*	1.1776*
2	0.5277	0.5359	17	0.4623	0.4242
3	0.3613	0.3613	18	0.5900	0.6459
4	0.5355	0.9613	19	1.2347	1.1611
5	0.7274	0.8952	20	0.9906	0.8297
6	1.0767	0.1234	21	0.8839	0.9832
7	0.8483	0.7681	22	1.4169	0.2770
8	0.4289	0.8835	23	0.5884*	0.5884*
9	0.1288*	0.1288*	24	1.2253	0.3161
10	1.1945	1.0802	25	0.7118	0.7744
11	0.4975	0.5164	26	0.4408	0.4408
12	0.2600*	0.6290	27	0.2009*	0.2009*
13	0.7829	0.5770	28	0.6968*	0.7828
14	0.9201	0.8967	29	1.0928	0.3435
15	0.9977	1.0328	30	0.9315	0.6683

From <Table 1>, MLE's of the parameters in BVW model are given by $\hat{\xi}_1 = 0.9373$, $\hat{\xi}_2 = 1.2224$, $\hat{\xi}_3 = 0.2491$ and m_1, m_2, m_3, m_4 are given by 11, 13, 2, and 4, respectively.

The values of test statistics and p-values are given by <Table 2>.

<Table 2> The values of test statistics and p-values

Test statistics / Values	$\phi_{MLE}(X, Y)$	$\phi_{M_i}(X, Y)$	$\pi_{MLE}(X, Y)$	$\pi_{M_i}(X, Y)$
The value of test statistics	10.0443	1.4638	4.4940	0.2894
p-value	0.0000	0.0716	0.0000	0.3861

From <Table 2>, we reject $H_o: \xi_3 = 0$ and $H_o: \xi_1 = \xi_2$ for independence and symmetry based on test statistics $\phi_{MLE}(X, Y)$ and $\pi_{MLE}(X, Y)$ with significance level 0.001. But we can not reject $H_o: \xi_3 = 0$ and $H_o: \xi_1 = \xi_2$ for independence and symmetry based on test statistics $\phi_{M_i}(X, Y)$ and $\pi_{M_i}(X, Y)$ with significance level 0.05.

Hence, we note that test statistics $\phi_{MLE}(X, Y)$ and $\pi_{MLE}(X, Y)$ based on MLE's perform better than $\phi_{M_i}(X, Y)$ and $\pi_{M_i}(X, Y)$ based on

m_1, m_2, m_3, m_4.

Hence, we note that estimates based on MLE's perform better than those based on natural estimates, more or less.

In our discussions, we have concentrated on the bivariate weibull model case. But we can apply our results, for a more general model.

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