

Recurrence Formula for the Central Moments of Number of Successes with n Poisson Trials

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Abstract

A sequence of n Bernoulli trials which violates the constant success probability assumption is termed as "Poisson trials". In this paper, the recurrence formula for the γ -th central moment of number of successes with n Poisson trials is derived. Romanovsky's method, based on the differentiation of characteristic function, is used in the derivation of recurrence formula for the central moments of conventional binomial distribution. Romanovsky's method is applied to that of Poisson trials in this paper. Some central moment calculation results are given to compare the central moments of Poisson trials with those of conventional binomial distribution.

Keywords : Bernoulli trials, Central moments, Conventional binomial distribution, Poisson trials, Romanovsky's method

1. Introduction

The Bernoulli trials are elemental to many discrete distributions including the conventional binomial distribution. Three assumptions underlie the conventional binomial distribution. Given a sequence of n Bernoulli trials, they are;

- ① Each Bernoulli trial is classified as 1 under "success" and 0 under "failure",
- ② Probability of "success" is constant,
- ③ Bernoulli trials are independent.

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Recently, researchers have moved to generalize the conventional binomial distribution by exploring the implications of the violation of two conventional assumptions ② and/or ③(Altham(1978), Drezner and Farnum(1993), Madsen(1993), Ng(1989), and Paul(1985, 1987)). Fu and Sproule(1995) derive the recurrence formula for the r -th moment of conventional binomial distribution which violates assumption ① using Romanovsky's reduction formula.

The easiest modification is to allow the probability of "success" to vary from trial to trial (violation of assumption ②), while retaining the assumption of independence between trials (assumption ③). In this setting with probability p_i of "success" at trial i ($i = 1, 2, 3, \dots, n$), the underlying trials are called "Poisson trials"(Feller 1968, p.218). With Poisson trials, the mean and variance of the number of successes ($= Y_n$) in n trials are given as follows(Feller 1968, p.231);

$$\mu = E(Y_n) = n\bar{p}, \quad \sigma^2 = Var(Y_n) = n\bar{p}(1 - \bar{p}) - ns^2, \quad (1.1)$$

$$\text{where } \bar{p} = \frac{\sum_{i=1}^n p_i}{n} \quad \text{and} \quad s^2 = \frac{\sum_{i=1}^n (p_i - \bar{p})^2}{n}.$$

In this paper, the recurrence formula for the r -th central moment ($r = 1, 2, 3, \dots$) of the number of successes in n Poisson trials is derived using the characteristic function.

2. Derivation of recurrence formula

Romanovsky(1923) proposes the interesting recurrence relation connecting the central moments of the conventional binomial distribution with parameters n and p . That is, the r -th central moment ($= \mu_r$) of that distribution is given by the following well-known recurrence formula;

$$\mu_r = pq \left\{ n(r-1) \mu_{r-2} + \frac{d\mu_{r-1}}{dp} \right\}, \quad r = 1, 2, 3, \dots$$

Using the above formula, we can get $\mu_1 = 0$, $\mu_2 = npq$, $\mu_3 = npq(q-p)$, etc.

In this section, the recurrence relation for the central moments of the number of successes in n Poisson trials is derived using Romanovsky's method. Let

$Y_n = \sum_{i=1}^n X_i$, where each X_i follows Bernoulli distribution with success probability p_i and X_i 's are mutually independent. The random variable Y_n ,

which violates the second assumption of conventional binomial random variable, is the sum of n Poisson trials. To derive the recurrence relation for the central moments of Y_n by Romanovsky's method, it is necessary to find the characteristic function of Y_n .

The characteristic function(c.f.) of Y_n referred to the mean as origin(that is, c.f. of $Y_n - E(Y_n) = Y_n - \mu$) is given by;

$$\begin{aligned} \phi_{(Y_n - \mu)}(t) &= E \{ \exp [it(Y_n - \mu)] \} \\ &= \exp(-it\mu) E \left\{ \exp(it \sum_{i=1}^n X_i) \right\} \\ &= \exp(-\theta n \bar{p}) \prod_{i=1}^n \{ q_i + p_i \exp(\theta) \}, \end{aligned} \tag{2.1}$$

where $\theta = it$ and $i = \sqrt{-1}$. Differentiate (2.1) with respect to θ . Then,

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{\mu_i \theta^{i-1}}{(i-1)!} &= \exp(-\theta n \bar{p}) \cdot \left\{ (-n \bar{p}) \prod_{i=1}^n \{ q_i + p_i \exp(\theta) \} \right. \\ &\quad \left. + \sum_{i=1}^n p_i \exp(\theta) \left(\prod_{j \neq i}^n \{ q_j + p_j \exp(\theta) \} \right) \right\} \\ &= \exp(-\theta n \bar{p}) \cdot \left\{ (\exp(\theta) - 1) \sum_{i=1}^n p_i q_i \left(\prod_{j \neq i}^n \{ q_j + p_j \exp(\theta) \} \right) \right\}, \end{aligned}$$

where μ_i is the i -th central moment of Y_n , which implies

$$\exp(\theta n \bar{p}) \cdot \left\{ \sum_{i=1}^{\infty} \frac{\mu_i \theta^{i-1}}{(i-1)!} \right\} = \{ \exp(\theta) - 1 \} \cdot \sum_{i=1}^n p_i q_i \left(\prod_{j \neq i}^n \{ q_j + p_j \exp(\theta) \} \right). \tag{2.2}$$

Expanding both sides of (2.2) yields the following;

$$\begin{aligned} &\left\{ 1 + \theta n \bar{p} + \frac{1}{2!} (\theta n \bar{p})^2 + \frac{1}{3!} (\theta n \bar{p})^3 + \dots \right\} \cdot \left\{ \mu_2 \theta + \frac{\mu_3 \theta^2}{2!} + \frac{\mu_4 \theta^3}{3!} + \dots \right\} \\ &= \left(\theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \dots \right) \cdot \left\{ \sum_{i=1}^n p_i q_i \left[\prod_{j \neq i}^n \left(1 + p_j \left(\theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \dots \right) \right) \right] \right\} \end{aligned} \tag{2.3}$$

{ = (1) · (2),

where (1) and (2) are the first and second terms of RHS of (2.3).

Now, compare the coefficient of θ^{r-1} on both sides of (2.3).

| | | |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------|---|---|
| L | H | S |
| $= \frac{1}{(r-1)!} \mu_r + \frac{n \bar{p}}{(r-2)!} \mu_{r-1} + \frac{(n \bar{p})^2}{2!(r-3)!} \mu_{r-2} + \dots + \frac{(n \bar{p})^{r-2}}{(r-2)!} \mu_2,$ | | |
| (2.4) | | |

RHS:

(i) Coefficient of $\theta^{(r-1)-k}$ in (1)

$$= \frac{1}{[(r-1)-k]!} \cdot \tag{2.5}$$

(ii) Coefficient of θ^k in (2)

$$= \sum_{i=1}^n p_i q_i \left\{ \sum_B \left(\prod_{j \neq i}^n \frac{p_j^{[(k+a_j)/(k+1)]}}{a_j!} \right) \right\}, \quad (2.6)$$

where (hereafter) the summand B extends over all non-negative values of a_j 's ($j=1, 2, \dots, n; j \neq i$) such that $\sum_{j \neq i} a_j = k$, and where $[(k+a_j)/(k+1)]$ denotes the greatest integer that does not exceed $(k+a_j)/(k+1)$.

From (2.5) and (2.6), coefficient of θ^{r-1} in RHS is given by

$$\sum_{k=0}^{r-2} \left\{ \frac{1}{[(r-1)-k]!} \sum_{i=1}^n p_i q_i \left[\sum_B \left(\prod_{j \neq i}^n \frac{p_j^{[(k+a_j)/(k+1)]}}{a_j!} \right) \right] \right\}. \quad (2.7)$$

By equating (2.4) and (2.7), we have

$$\begin{aligned} & \frac{1}{(r-1)!} \mu_r + \frac{\bar{n}\bar{p}}{(r-2)!} \mu_{r-1} + \frac{(\bar{n}\bar{p})^2}{2!(r-3)!} \mu_{r-2} + \dots + \frac{(\bar{n}\bar{p})^{r-2}}{(r-2)!} \mu_2 \\ &= \sum_{k=0}^{r-2} \left\{ \frac{1}{[(r-1)-k]!} \sum_{i=1}^n p_i q_i \left[\sum_B \left(\prod_{j \neq i}^n \frac{p_j^{[(k+a_j)/(k+1)]}}{a_j!} \right) \right] \right\}. \end{aligned} \quad (2.8)$$

Finally, multiplying $(r-1)!$ on both sides of (2.8) results in the final recurrence relation for the r -th central moment of Y_n . Note that $\mu_1 = 0$.

$$\begin{aligned} & \mu_r + \sum_{i=1}^{r-1} \binom{r-1}{i} (\bar{n}\bar{p})^i \mu_{r-i} \\ &= \sum_{k=0}^{r-2} \left\{ \frac{(r-1)!}{[(r-1)-k]!} \right. \\ & \left. \sum_{i=1}^n p_i q_i \left[\sum_B \left(\prod_{j \neq i}^n \frac{p_j^{[(k+a_j)/(k+1)]}}{a_j!} \right) \right] \right\}, \quad r=2, 3, 4, \dots \end{aligned}$$

3. Calculation and Comparison

In this section, the r -th central moment of some Poisson trials is calculated and compared with that of conventional binomial distribution. Three groups of p_i 's are formulated according to the value of \bar{p} to include various cases as possible. The first group consists of four p_i 's sets with $\bar{p} = 0.50$. The second and third one contain three p_i 's sets each with $\bar{p} = 0.30$ and $\bar{p} = 0.70$, respectively. For all groups, n is set to 10 since it is found (not included in this work) that the comparison results do not depend on n . [Table 1] includes ten

p_i 's sets used in the calculation. The first set(set (a)) in each group consists of equal p_i 's, so it corresponds to the conventional binomial distribution. The remaining seven p_i 's sets are Poisson trials. Among them, set (b) of each group corresponds to uniformly distributed p_i 's, set (c) contains p_i 's skewed to left, and set (d) skewed to right.

The central moment(upto $r = 10$) calculation results for ten p_i 's sets are given

[Table 1]: p_i 's sets used in calculation

| Group | \bar{p} | p_i 's | Description |
|-------|-----------|--------------------------------------------------------|---------------------------------|
| I | 0.50 | (a) 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5 | Equal p_i 's |
| | | (b) 0.3, 0.3, 0.4, 0.4, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7 | Different p_i 's, symmetric |
| | | (c) 0.1, 0.15, 0.15, 0.6, 0.6, 0.6, 0.7, 0.7, 0.7, 0.7 | Diff. p_i 's, skewed to left |
| | | (d) 0.3, 0.3, 0.3, 0.3, 0.4, 0.4, 0.4, 0.85,0.85, 0.9 | Diff. p_i 's, skewed to right |
| II | 0.30 | (a) 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3 | Equal p_i 's |
| | | (b) 0.1, 0.1, 0.2, 0.2, 0.3, 0.3, 0.4, 0.4, 0.5, 0.5 | Different p_i 's, symmetric |
| | | (d) 0.1, 0.1, 0.1, 0.1, 0.1, 0.2, 0.2, 0.5, 0.7, 0.9 | Diff. p_i 's, skewed to right |
| III | 0.70 | (a) 0.7, 0.7, 0.7, 0.7, 0.7, 0.7, 0.7, 0.7, 0.7, 0.7 | Equal p_i 's |
| | | (b) 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.8, 0.8, 0.9, 0.9 | Different p_i 's, symmetric |
| | | (c) 0.1, 0.3, 0.5, 0.8, 0.8, 0.9, 0.9, 0.9, 0.9, 0.9 | Diff. p_i 's, skewed to left |

[Table 2] Central moment calculation results

| Set r | Group I | | | | Group II | | | Group III | | |
|------------|---------|---------|--------|--------|----------|--------|---------|-----------|---------|--------|
| | (a) | (b) | (c) | (d) | (a) | (b) | (d) | (a) | (b) | (c) |
| 2 | 2.5 | 2.30 | 1.905 | 1.905 | 2.1 | 1.9 | 1.32 | 2.1 | 1.9 | 1.32 |
| 3 | 0 | 0 | -0.23 | 0.23 | 0.84 | 0.60 | 0.40 | -0.84 | -0.60 | -0.40 |
| 4 | 17.5 | 14.98 | 10.45 | 10.45 | 12.68 | 10.36 | 5.31 | 12.68 | 10.36 | 5.31 |
| 5 | 0 | 0 | -3.69 | 3.69 | 16.36 | 10.78 | 5.16 | -16.36 | -10.78 | -5.16 |
| 6 | 190 | 152.95 | 92.05 | 92.05 | 128.8 | 93.43 | 37.31 | 128.76 | 93.43 | 37.31 |
| 7 | 0 | 0 | -59.46 | 59.46 | 320.2 | 193.9 | 70.92 | -320.2 | -193.9 | -70.92 |
| 8 | 2680 | 2049.8 | 1090.2 | 1090.2 | 1904.8 | 1210.7 | 391.4 | 1904.8 | 1210.7 | 391.4 |
| 9 | 0 | 0 | -1044 | 1044 | 6970.3 | 3859.1 | 1130.3 | -6970 | -3859.1 | -1130 |
| 10 | 44983.8 | 33000.1 | 15876 | 15876 | 37634.2 | 20918 | 5608.23 | 37634.2 | 20918 | 5608.2 |

in [Table 2]. As can be seen in the table, variance($r = 2$) of the sum of n Poisson trials(sets (b), (c), (d) of each group) is smaller than that of conventional binomial

distribution(set (a) of each group) for all groups. That is, variance of each p_i 's set becomes smaller as p_i 's get more variable in each set, which coincide with the result given in (1.1). Although it is difficult to derive a neat formula as (1.1) for r greater than 2, it is clear from [Table 2] that the even r -th central moment of Poisson trials is smaller than that of conventional binomial distribution for all groups, if n and \bar{p} of Poisson trials are the same as the number of Bernoulli trials and success probability of conventional binomial distribution respectively. No clear comparison results on the magnitude of odd r -th central moment can be found in [Table 2].

4. Conclusion

The recurrence formula for the r -th central moment of number of successes with n Poisson trials is derived using Romanovsky's method, which is based on the differentiation of characteristic function of Y_n referred to the mean as origin. The central moment calculation results show that the even r -th central moment of Poisson trials is smaller than that of conventional binomial distribution if n and \bar{p} of Poisson trials are the same as the number of Bernoulli trials and success probability of conventional binomial distribution. For the odd r -th central moment comparison, no clear conclusions can be made.

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