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ON STABILITY OF A QUADRATIC FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the new quadratic type functional equation

$$f(2x+y) - f(x+2y) = 3f(x) - 3f(y)$$

and prove the stablility of this equation in the spirit of Hyers, Ulam, Rassias and Găvruţa.

1. Introduction

In 1940, S. M. Ulam[21] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?.

In the other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. If we turn our attention to the case of functional equations, we can ask the question: When

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the solutions of an equation differing slightly from a given one must be close to the true solution of the given equation ?

The quadratic function $f(x) = cx^2$ $(x \in \mathbb{R})$ satisfies the following equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

Hence, it is natural that the above equation is called the quadratic functional equation. In particular, every solution of the quadratic equation is said to be a quadratic function. It is well-known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x,x)$ for all x [1, 15]. The bi-additive function B is given by

$$B(x,y) = \frac{1}{4}(f(x+y) - f(x-y)).$$

The Hyers-Ulam stability problem for the quadratic functional equation was proved by F. Skof [20] for function $f : E_1 \to E_2$, where E_1 is normed space and E_2 a Banach space. P. W. Cholewa [4] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an abelian group. S. Czerwik[5] proved the Hyers-Ulam-Rassias stability of the follwing functional equation.

> Let E_1 be a normed space and E_2 a Banach space and let $\eta \geq 0$ and $p > 2$ be given real numbers. Let $f : E_1 \to E_2$ satisfy the condition $||f(x+y)+f(x-y)-2f(x)-2f(y)|| \leq \eta(||x||^p + ||y||^p)$ for all $x \in E_1$. Then there exists unique quadratic mapping $h : E_1 \to E_2$ such that $||h(x) - f(x)|| \leq 2(2^p - 4)^{-1}\eta||x||^p$ for all $x \in E_1$.

A. Grabiec [9] has generalized these results mentioned above. Recently, the Hyers-Ualm-Rassias stability of the quadratic equation has been extensively investigated (see [11, 12, 13]).

Consider the following functional equations :

$$f(2x+y)+f(x+2y)=4f(x+y)+f(x)+f(y) \tag{1.2}$$

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+6f(x) \tag{1.3}$$

$$f(2x+y)-f(x+2y)=3f(x)-3f(y) \tag{1.4}$$

The above equation (1.2) is the other form of quadratic equation. H. M. Kim [3] showed that (1.2) and (1.3) is equivalent to the original quadratic equation (1.1). In this paper we investigate the new quadratic type functional equation (1.4) and prove the stability of this equation in the spirit of Hyers, Ulam, Rassias and Găvruţa.

By $\mathbb{N}$ and $\mathbb{R}$ we denote the set of positive integers and of real numbers, respectively.

2. Solutions of (1.4)

In this section, we will show that the general solution of (1.4) is $Q(x)+C$, where Q is quadratic and C is a constant in Y.

THEOREM 2.1. *Let X and Y be real vector spaces. A function $f: X \to Y$ satisfies the functional equation (1.4) if and only if $f(x) = B(x,x)+C$ for some symmetric bi-additive function $B: X \times X \to Y$ and some C in Y.*

Therefore every solution f of functional equation (1.4) with $f(0)=0$ is also a quadratic function.

Proof. Let $f: E_1 \to E_2$ satisfy the functional equation (1.4). If we put $g(x) = f(x) - f(0)$, we obtain that g is also a solution of (1.4) and $g(0) = 0$. So we may assume without loss of generality that f is a solution of (1.4) and $f(0) = 0$.

If we put $y = -x$ in (1.4), we get $f(-x) = f(x)$. By putting $y = 0$ and $y = -2x$ in (1.4), we see that $4f(x) = f(2x)$ and $9f(x) = f(3x)$, respectively.

Replacing x, y by $x+y$, $x-\frac{y}{2}$ and also x, y by $x-y$, $x+\frac{y}{2}$ in (1.4), respectively, we obtain that

$$f(x+y)-f(x-\frac{y}{2})=\frac{1}{3}f(3(x+\frac{y}{2}))-\frac{1}{3}f(3x) \tag{2.1}$$

$$f(x-y)-f(x+\frac{y}{2})=\frac{1}{3}f(3(x-\frac{y}{2}))-\frac{1}{3}f(3x). \tag{2.2}$$

Adding (2.1) to (2.2), we can get

$$f(x+y)+f(x-y)+\frac{2}{3}f(3x)=4f(x+\frac{y}{2})+4f(x-\frac{y}{2}), \tag{2.3}$$

which implies that

$$f(x+y)+f(x-y)+6f(x)=f(2x+y)+f(2x-y)$$

According to [3], the above equation is equivalent to $f(x+y)+f(x-y)=2f(x)+2f(y)$. Hence $f(x)=B(x,x)$ for some symmetric biadditive function.

Conversly, suppose that there exist a symmetric bi-additive function $B:X\times X\to Y$ and an element C in Ysuch that $f(x)=B(x,x)+C$ for all $x\in X$. Then it follows easily that f is a solution of the equation (1.4).

3. Hyers-Ulam-Rassias Stability of (1.4)

Throughout this section X and Y will be a real normed linear space and a real Banach space, respectively. Given by $f:X\to Y$, we set

$$Df(x,y)=f(2x+y)-f(x+2y)-3f(x)+3f(y) \tag{3.1}$$

for all $x,y\in X$

We denote by $\varphi : X \times X \to [0, \infty)$ a mapping such that either

$$\Phi_1(x,y) := \sum_{k=0}^{\infty} 4^{-k}\varphi(2^k x, 2^k y) < \infty \qquad (a)$$

or

$$\Phi_2(x,y) := \sum_{k=1}^{\infty} 4^{k}\varphi(\frac{x}{2^k}, \frac{y}{2^k}) < \infty \qquad (b)$$

for all $x, y \in X$.

THEOREM 3.1. *Assume that a function $f : X \to Y$ satisfies*

$$||Df(x,y))|| \leq \varphi(x,y) \tag{3.2}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying (1.4) such that

$$||f(x) - Q(x) - f(0)|| \leq \frac{1}{4}\Phi_1(x,0) \ if \ \varphi \ satisfies \ (a), \tag{3.3}$$

$$||f(x) - Q(x)|| \leq \frac{1}{4}\Phi_2(x,0) \ if \ \varphi \ satisfies \ (b) \tag{3.4}$$

for all $x \in X$.

If, moreover, f is measurable or $\mathbb{R} \ni t \to f(tx)$ is continuous for each fixed $x \in X$, then Q satisfies the condition $Q(tx) = r^2 Q(x)$ for all $x \in X$ and $r \in \mathbb{R}$.

Proof. Let $g : X \to Y$ be the function defined by $g(x) = f(x) - f(0)$. Then $g(0) = 0$ and

$$\begin{aligned} ||Df(x,y))|| = ||Dg(x,y))|| &= ||g(2x+y) - g(x+2y) - 3g(x) + 3g(y)|| \\ &\leq \varphi(x,y) \end{aligned} \tag{3.5}$$

for all $x \in X$.

Putting $y = 0$ in (3.5) yields

$$||g(2x) - 4g(x)|| \leq \varphi(x,0). \tag{3.6}$$

Thus

$$||\frac{g(2x)}{4} - g(x)|| \le \frac{1}{4}\varphi(x,0). \tag{3.7}$$

Assume that φ satisfies the condition (a).

Applying the induction argument to n we obtain

$$||\frac{g(2^n x)}{4^n} - g(x)|| \le \frac{1}{4}\sum_{k=0}^{n-1} 4^{-k}\varphi(2^k x, 0). \tag{3.8}$$

Hence, using (3.7), we see that

$$\begin{aligned} ||\frac{g(2^n x)}{4^n} - \frac{g(2^m x)}{4^m}|| &= \frac{1}{4^m}||\frac{g(2^{n-m}2^m x)}{4^{n-m}} - g(2^m x)|| \\ &\le \frac{1}{4}\sum_{k=0}^{n-m-1} 4^{-k-m}\varphi(2^{k+m}x, 0) \end{aligned}$$

for all $n, m \in \mathbb{N}$ with $n > m$ and $x \in X$. This shows that for any $x \in X$ $\{\frac{g(2^n x)}{4^n}\}$ is a Cauchy sequence in Y and thus converges. Therefore we can define a function $Q : X \to Y$ by

$$Q(x) = \lim_{x\to\infty}\frac{g(2^n x)}{4^n} = \lim_{x\to\infty}\frac{f(2^n x)}{4^n}$$

for all $x \in X$.

Substituting $2^n x$ and $2^n y$ for x and y in (3.2), we have

$$\begin{aligned} ||f(2^{n+1}x + 2^n y) - f(2^n x + 2^{n+1}y) &- 3f(2^n x) + 3f(2^n y)|| \\ &\le \varphi(2^n x, 2^n y) \end{aligned} \tag{3.9}$$

Dividing both sides by 4^n and taking the limit as $n \to \infty$ in (3.9) we find that Q satisfies the equation

$$Q(2x + y) - Q(x + 2y) - 3Q(x) + 3Q(y) = 0. \tag{3.10}$$

Since $Q(0) = 0$, Q is a quadratic function by Theorem 2.1. Taking the limit in (3.8) as $n \to \infty$, we obtain

$$||Q(x) - g(x)|| \leq \frac{1}{4}\Phi_1(x, 0) \tag{3.11}$$

for all $x \in X$. Consequently, we know that (3.3) holds.

If Q' is another quadratic function satisfying (3.11), then we have

$$\begin{aligned} ||Q(x) - Q'(x)|| &\leq \frac{1}{4^n}||Q(2^n x) - Q'(2^n x)|| \\ &\leq \{||Q(2^n x) + f(0) - f(2^n x)|| + ||f(2^n x) - f(0) - Q'(2^n x)||\} \\ &\leq \frac{1}{4^n}\frac{\Phi_1(2^n x, 0)}{2} \longrightarrow \quad 0 \quad as \quad n \to \infty. \end{aligned}$$

for all $x \in X$. Therefore we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. Assume that φ satisfies the condition (b). Then we easily see that $f(0) = 0$ and $f = g$ since the series (b) converges for $x = 0$.

Replacing x by $\frac{x}{2}$ in (3.9) we get

$$||f(x) - 4f(\frac{x}{2})|| \leq \varphi(\frac{x}{2}, 0)$$

for all $x \in X$.

Applying the induction argument to n we obtain

$$||f(x) - 4^n f(\frac{x}{2^n})|| \leq \frac{1}{4}\sum_{k=1}^{n} 4^k \varphi(2^{-k}x, 0) \tag{3.12}$$

Hence the following inequality

$$||4^n f(\frac{x}{2^n}) - 4^m f(\frac{x}{2^m})|| \leq \frac{1}{4}\sum_{k=1}^{n-m} 4^{k+m}\varphi(2^{-k-m}x, 0)$$

holds for all $n, m \in \mathbb{N}$ with $n > m$ and $x \in X$. This shows that for any $x \in X$ $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y and thus converges.

Therefore we can define a function $Q : X \to Y$ by

$$Q(x) = \lim_{x\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$.

Substituting $2^{-n}x$ and $2^{-n}y$ for x and y in (3.2)

$$\begin{aligned} ||f(2^{-n+1}x + 2^{-n}y) - f(2^{-n}x + 2^{-n+1}y) &- 3f(2^{-n}x) + 3f(2^{-n}y)|| \\ &\le \varphi(2^{-n}x, 2^{-n}y). \end{aligned}$$

Multiplying on both sides by 4^n and Taking the limit as $n \to \infty$ in the above inequality we find that Q satisfies the equation

$$Q(2x + y) - Q(x + 2y) - 3Q(x) + 3Q(y) = 0. \tag{3.13}$$

Thus Q is quadratic. Taking the limit in (3.12) as $n \to \infty$, we obtain

$$||Q(x) - f(x)|| \le \frac{1}{4}\Phi_2(x, 0) \tag{3.14}$$

for all $x \in X$. Consequently, we know that (3.4) holds.

The rest of the proof is similar to the case (a).

From the main Theorem 3. 1, we obtain the following corollary concerning the stability of the equation (1.4)

Corollary 3.2. *Let $p > 0$, $\theta \ge 0$, and $\epsilon > 0$ be real numbers such that $\theta = 0$ if $p > 2$. Assume that a function $f : X \to Y$ satisfies*

$$||Df(x, y))|| \le \theta + \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying (1.4) such that in case $0 < p < 2$

$$||f(x) - Q(x) - f(0)|| \le \frac{1}{3}\theta + \frac{1}{4 - 2^p}\epsilon||x||^p$$

for all $x \in X$, while in the case $p > 2$

$$||f(x) - Q(x)|| \le \frac{1}{2^p - 4}\epsilon||x||^p$$

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.3. *Assume that a function $f : X \to Y$ satisfies*

$$||Df(x,y))|| \le \theta$$

for all $x, y \in X$. then there exists a unique quadratic function $Q : X \to Y$ satisfying (1.4) such that

$$||f(x) - Q(x) - f(0)|| \le \frac{\theta}{3}$$

for all $x \in X$

Let X be a normed linear space and define $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\varphi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi_0(\lambda) > 0 \quad \text{for all} \quad \lambda > 0,$$

$$\varphi_0(2) < 2,$$

$$\varphi_0(2\lambda) \le \varphi_0(2)\varphi_0(\lambda) \quad \text{for all} \quad \lambda > 0$$

$$H(\lambda t, \lambda s) \le \varphi_0(\lambda) H(t,s) \quad \text{for all} \quad t, s \in \mathbb{R}_+, \quad \lambda > 0.$$

We consider the following in the next corollary. Let

$$\varphi(x,y) := H(||x||, ||y||).$$

Then

$$\begin{aligned} \varphi(2^k x, 2^k y) &= H(2^k ||x||, 2^k ||y||) \\ &\le \varphi_0(2^k) H(||x||, ||y||) \\ &\le (\varphi_0(2))^k H(||x||, ||y||) \end{aligned}$$

and because $\varphi_0(2) < 2$ we have

$$\begin{aligned}\Phi_1(x,y) &\leq \sum_{k=0}^{\infty} 4^{-k}(\varphi_0(2))^k H(|||x||,||y||) \\ &= \frac{1}{1-\varphi_0(2)/4} H(||x||,||y||).\end{aligned}$$

Hence, we can see that the following corollary holds.

COROLLARY 3.4. *Assume that a function $f : X \to Y$ satisfies*

$$||Df(x,y))|| \leq H(||x||,||y||)$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying (1.4) such that

$$||f(x) - Q(x) - f(0)|| \leq \frac{H(||x||,0)}{4-\varphi_0(2)}$$

for all $x \in X$

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