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FLOWS INDUCED BY COVERING MAPS

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Abstract. The purpose of this paper is to prove flow induced by a covering map. Lee and Park had studied semiflows induced by a covering map in 1997 [1]. This proof differs from the proof of Lee and Park. Notice that for the proof of this paper, we use the fact that $\mathbb{R}$ is connected space.

K.B. Lee and J.S. Park had studied semiflows induced by a covering map in 1997 [1]. In this paper, we shall have the same result for flow and covering map. Notice that for the proof of this paper, we use the fact that $\mathbb{R}$ is connected space. A *flow* in the space X is a function $q = \phi(p, t)$ which assigns to each point p of the space X and to each real number t $(-\infty < t < \infty)$ a definite point$q \in X$ and possesses the following three properties; (1) Initial conditions: $\phi(p, 0) = p$ for any point $p \in X$. (2) Group property: $\phi(\phi(p, t_1), t_2) = \phi(p, t_1 + t_2)$ for any point $p \in X$ and any real t_1 and t_2. (3) Continuous condition: map $\phi : X \times \mathbb{R} \to X$ is continuous.

Theorem 1. *A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.*

Proof. See [2] □

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THEOREM 2. *For any flow* (X, ϕ) *and a covering map* $P\colon \overline{X} \to X$, *there exists a unique flow* $(\overline{X}, \overline{\phi})$ *such that* $P(\overline{\phi}(\overline{x}, t)) = \phi(P(\overline{x}), t)$ *for all* $\overline{x} \in \overline{X}$ *and* $t \in \mathbb{R}$.

Proof. Let $\phi : X \times \mathbb{R} \to X$ be a flow and let $P : \overline{X} \to X$ be a covering map.

For all $\overline{x} \in \overline{X}$ and $t \in \mathbb{R}$, put $x \equiv p(\overline{x})$. Define a map $\alpha(s) = \phi(x, st)$ for $s \in I = [0, 1]$. Then α is a path in X from x to $\phi(x, t)$. Let $\overline{\alpha} : I \to \overline{X}$ be a lifting of α beginning at $\overline{x}$.

Define a map $\overline{\phi} : \overline{X} \times \mathbb{R} \to \overline{X}$ by $\overline{\phi}(\overline{x}, t) = \overline{\alpha}(1)$. Then from

$$P(\overline{\phi}(\overline{x}, t)) = P(\overline{\alpha}(1)) = \alpha(1) = \phi(x, t) = \phi(P(\overline{x}), t)$$

, we obtain $P \circ \overline{\phi} = \phi \circ P$. Also, the uniqueness of $\overline{\phi}$ is clear. First, to prove conditions of flow, we prove that $\overline{\alpha * \beta} = \overline{\alpha} * \overline{\beta}$. Let $\overline{x} \in \overline{X}$ and $t, u \in \mathbb{R}$. Define a map $\alpha : I \to X$ by $\alpha(s) = \phi(x, st)$. Then α is a path in X from x to $\phi(x, t)$. Let $\overline{\alpha} : I \to \overline{X}$ be a lifting of α beginning at $\overline{x}$. Define a map $\beta : I \to X$ by $\beta(s) = \phi(\phi(x, t), su)$. Then β is a path in X from $\phi(x, t)$ to $\phi(\phi(x, t), u) = \phi(x, t + u)$. Let a map $\overline{\beta} : I \to \overline{X}$ be the lifting of β beginning at $\overline{\phi}(\overline{x}, t)$. Then we obtain $\overline{\phi}(\overline{\phi}(\overline{x}, t), u) = \overline{\beta}(1)$.

Define a map $\gamma : I \to X$ by $\gamma(s) = \phi(x, s(t + u))$. Then γ is a path in X from x to $\phi(x, t + u)$.

Let a map $\overline{\gamma} : I \to \overline{X}$ be the lifting of γ beginning at $\overline{x}$.

From the definition, we obtain $\overline{\phi}(\overline{x}, t + u) = \overline{\gamma}(1)$.

Let $\overline{\alpha * \beta} : I \to \overline{X}$ be the lifting of $\alpha * \beta$ beginning at $\overline{x}$. Then, $\overline{\alpha} * \overline{\beta}(0) = \overline{\alpha}(0) = \overline{x}$. Also, if $0 \le s \le \frac{1}{2}$, then $P(\overline{\alpha} * \overline{\beta}(s)) = P(\overline{\alpha}(2s)) = \alpha(2s)$ and if $\frac{1}{2} \le s \le 1$, $P(\overline{\alpha} * \overline{\beta}(s)) = P(\overline{\beta}(2s - 1)) = \beta(2s - 1)$. Therefore, $P \circ (\overline{\alpha} * \overline{\beta}) = \alpha * \beta$. From the uniqueness of lifting, $\overline{\alpha * \beta} = \overline{\alpha} * \overline{\beta}$. Initial condition: $\overline{\phi}(\overline{x}, 0) = \overline{x}$ for any point $\overline{x} \in \overline{X}$.

To prove Group property,
define a map $H : I \times I \to X$ by

$$H(s,r) = \begin{cases} \phi(x,((2-r)t+ru)s), & \text{for } s \le \frac{1}{2} \\ \phi(x,(rt+(2-r)u)s+(1-r)(t-u)), & \text{for } s \ge \frac{1}{2}. \end{cases}$$

Then H is a continuous map. If $0 \le s \le \frac{1}{2}$, then $H(s,0) = \phi(x,2st) = \alpha(2s)$ and if $\frac{1}{2} \le s \le 1$, then $H(s,o) = \phi(x,u(2s-1)+t) = \phi(\phi(x,t),\,(2s-1)u) = \beta(2s-1)$.

Therefore, we have $H(s,0) = \alpha * \beta(s)$, $H(s,1) = \phi(x,s(t+u)) = v(s)$, $H(0,r) = \phi(x,0) = x$, $H(1,r) = \phi(x,t+u)$. Hence, H is a path homopty between $\alpha * \beta$ and γ. By Lemma 54.2 [2], $\overline{\alpha * \beta}(1) = \overline{\gamma}(1)$.

Since $\overline{\alpha * \beta}(1) = \overline{\alpha} * \overline{\beta}(1) = \overline{\beta}(1) = \overline{\phi}(\overline{\phi}(\overline{x},t),u)$ and $\overline{\gamma}(1) = \overline{\phi}(\overline{x},t+u)$, we have $\overline{\phi}(\overline{\phi}(\overline{x},t),u) = \overline{\phi}(\overline{x},t+u)$.

Now we prove continuity of $\overline{\phi}$. Let $\overline{x} \in \overline{X}$. Put $M = \{t \in \mathbb{R} \mid \overline{\phi}$ is continuous at $(\overline{x},t)\}$. We claim $M = \mathbb{R}$. To prove this fact, we use connectedness of $\mathbb{R}$.

First , to show that M is a nonempy set, let U be an elementry neighborhood of $x \equiv p(\overline{x})$. Then $P^{-1}(U) = \cup V_i$. Let $\overline{x} \in V_k$ and $q \equiv P \mid V_k :\to U$. For any neighborhood $W \subset V_k$ of $\overline{x}$, $q(W)$ is a neighborhood of $x = \phi(x,0)$.

From continuity of ϕ at $(x,0)$, there exist a neighborhood $B \subset U$ of x and $\delta > 0$ such that $\phi(B \times (-\delta,\delta)) \subset q(W)$. Also, since P is continuous at $\overline{x}$, there a neighborhood $A \subset V_k$ of $\overline{x}$ such that $P(U) \subset B$. Choose $\overline{y} \in A$ and $t \in (-\delta,\delta)$.

From $y \equiv P(\overline{y}) \in P(A) \subset B$, we obtain $\phi(\{y\} \times (-\delta,\delta)) \subset \phi(B \times (-\delta,\delta)) \subset q(W)$.

Define a map $\alpha : I \to q(W)$ by $\alpha(s) = \phi(y,st)$. Then α is a path in $q(W)$ from y to $\phi(s,t)$. $\beta \equiv q^{-1} \circ \alpha : I \to W$ is a lifting of α beginning at $\overline{y}$.

Then $\overline{\phi}(\overline{x},t) = \beta(1) \in W$. Hence, we have $\overline{\phi}(A \times (-\delta,\delta)) \subset W$. Since $\overline{\phi}$ is continuous at $(\overline{x},0)$, we have $0 \in M$.

To show openness of M, let $t \in M$. Let U be an elementary neighborhood of $\phi(x,t)$.

Then $P^{-1}(U) = \cup V_i$. Let $\overline{\phi}(\overline{x},t) \in V_k$ and $q \equiv P \mid V_k : V_k \to U$.

Since $\overline{\phi}$ is continuous at $(\overline{x},t)$, there is a neighborhood A of $\overline{x}$ and $\delta > 0$ such that $\overline{\phi}(A \times (t-\delta,t+\delta)) \subset V_k$. Let $u \in (t-\delta,t+\delta)$.

Then, $\overline{\phi}(\overline{x},u) \in \overline{\phi}(A \times (t-\delta,t+\delta)) \subset V_k$ For any neighborhood $W \subset V_k$ of $\overline{\phi}(\overline{x},u)$, $q(W)$ is a neighborhood of $\phi(x,u)$.

Since ϕ is continuous at (x,u), there exists a neighborhood B of x and $0 < \epsilon < min\{t+\delta-u, u-t+\delta\}$ such that $\phi(B \times (u-\epsilon,u+\epsilon)) \subset q(W)$. Also, since P is continuous at $\overline{x}$, there exists a neighborhood $D \subset A$ of $\overline{x}$ such that $P(D) \subset B$.

Take $\overline{y} \in D$ and $v \in (u-\epsilon,u+\epsilon) \subset (t-\delta,t+\delta)$. Since $\overline{\phi}(\overline{y},v) \in \overline{\phi}(A \times (t-\delta,t+\delta)) \subset V_k$ and $P(\overline{y}) \in P(D) \subset B$, we have $q(\overline{\phi}(\overline{y},v)) = \phi(P(\overline{y}),v) \in \phi(B \times (u-\epsilon,u+\epsilon)) \subset q(W)$. Hence $\overline{\phi}(\overline{y},v) \in q^{-1}(q(W)) = W$.

From $\overline{\phi}(D \times (u-\epsilon,u+\epsilon)) \subset W$, $\overline{\phi}$ is continuous at $(\overline{x},u)$. Consequently, $u \in M$ and $(t-\delta,t+\delta) \subset M$, ending the proof of openness. Next we shall closedness of M. Take $t \in \overline{M}$.

Let U be an elementary neighborhood of $\phi(x,t)$.

Then $P^{-1}(U) = \cup V_i$. Let $\overline{\phi}(\overline{x},t) \in V_k$ and $q \equiv P \mid V_k : V_k \to U$.

For any neighborhood $W \subset V_k$ of $\overline{\phi}(\overline{x},u)$, $q(W)$ is a neighborhood of $\phi(x,u)$.

Since ϕ is continuous at (x,t), there exist a neighborhood B of x and $\delta > 0$ such that $\phi(B \times (t-\delta,t+\delta)) \subset q(W)$. Also, $(t-\delta,t+\delta) \cap M \neq \emptyset$. Take $u \in (t-\delta,t+\delta) \cap M$. Define a map $\alpha : I \to q(W)$ by $\alpha(s) = \phi(\phi(x,t), s(u-t)) = \phi(x,(1-s)t+su))$.

Then α is a path in $q(W)$ from $\phi(x,t)$ to $\phi(\phi(x,t),u-t)$. And

$\beta \equiv q^{-1} \circ \alpha : I \to W$ is a lifting of α beginning at $\overline{\phi}(\overline{x}, t)$. Also, we have $\overline{\phi}(\overline{x}, u) = \overline{\phi}(\overline{\phi}(\overline{x}, t), u - t) = \beta(1) \in W$.

From continuity of P at $\overline{x}$ and $\overline{\phi}$ at $(\overline{x}, u)$, there is a neighborhood A of $\overline{x}$ such that $P(A) \subset B$ and $\overline{\phi}(A \times \{u\})) \subset W$. Take $\overline{y}$ and $v \in (t - \delta, t + \delta)$. Since $\overline{\phi}(\overline{x}, u) \in \overline{\phi}(A \times \{u\})) \subset W$ and $y \equiv P(\overline{y}) \in P(A) \subset B$, we have $\phi(\{y\} \times (t - \delta, t + \delta)) \subset \phi(B \times (t - \delta, t + \delta)) \subset q(W)$.

Define a map $\alpha : I \to q(W)$ by $\alpha(s) = \phi(\phi(y, u), s(v - u)) = \phi(y, (1 - s)u + sv))$. Then α is a path in $q(W)$ from $\phi(y, u)$ to $\phi(\phi(y, u), v - u)$.

And $\beta \equiv q^{-1} \circ \alpha : I \to W$ is a lifting of α beginning at $\overline{\phi}(\overline{y}, t)$. Also, we have $\overline{\phi}(\overline{y}, u) = \overline{\phi}(\overline{\phi}(\overline{y}, t), v - u) = \beta(1) \in W$.

From $\overline{\phi}(A \times (t - \delta, t + \delta)) \subset W$, $\overline{\phi}$ is continuous at $(\overline{x}, t)$. Hence $t \in M$, ending closedness. Since $\mathbb{R}$ is connected, $M = \mathbb{R}$ by Theorem 1. Consequently, $\overline{\phi}$ is continuous.

□

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