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ON THE DYNAMICAL PROPERTIES OF SOME FUNCTIONS

Seung-Jae Yoo

Abstract. This note is concerned with some properties of fixed points and periodic points. First, we have constructed a generalized continuous function to give a proof for the fact that, as the reverse of the Sharkovsky theorem[16], for a given positive integer n, there exists a continuous function with a period-n point but no period-m points where m is a predecessor of n in the Sharkovsky ordering. Also we show that the composition of two transcendental meromorphic functions, one of which has at least three poles, has infinitely many fixed points.

1. Introduction

In the iteration theory, there are many important results for the fixed points and periodic points which are originated by so many mathematicians, notably P. Fatou[11], G. Julia[12], I.N. Baker[2,3] and R. Devaney[10], W. Bergweiler[5,6,7] etc, who have interested in this attractive subject insistently.

One of the remarkable results in this subject is the Sharkovsky theorem[16] published in 1964 that guarantees the existence of periodic points of periods-all successors of a given period in the Sharkovsky ordering. Also it is well-known that for a given positive integer n, there exists a continuous function with n-periodic points.

In [2], I.N. Baker showed that a transcendental entire function has periodic points of all periods except at most one integer, even though

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a transcendental entire function and a transcendental meromorphic function need not have fixed points[11]. And W. Bergweiler showed that the composite function $f \circ g$ of a transcendental entire function g and a transcendental meromorphic function f has infinitely many fixed points, and furthermore the same holds for the case of meromorphic function f which has at least two distinct poles[6].

In this note, we have constructed a generalized continuous function Φ to give a proof for the fact that, as the reverse of the Sharkovsky theorem, for a given positive integer n, there exists a continuous function with a period-n point but no period-m points where m is a predecessor of n in the Sharkovsky ordering(Theorem A).

Also we show that the composition of two transcendental meromorphic functions, one of which has at least three poles, has infinitely many fixed points(Theorem B).

2. Preliminaries

For a point x_0 in the domain of a given function f,

$$f^{[n]}(x_0) = f(f^{[n-1]}(x_0)), \quad f^{[0]}(x_0) = x_0$$

for $n = 1, 2, \cdots$ is called the *n-iterate x_0 for f*. If $f^{[n]}(x_0) = x_0$ and any two elements of the set $\{x_0, f(x_0), \cdots, f^{[n-1]}(x_0)\}$ are distinct, then this set is called an *n-cycle* and we say that x_0 has a *period-n* for f.

Now we note some worthful results for the periodic points. I.N Baker[2] proved that if a rational function f of degree $d \geq 2$ has no periodic points of period-n then (d, n) is one of the pairs

$$(2, 2), (2, 3), (3, 2), (4, 2).$$

Moreover, if f is a polynomial then the only case $(2, 2)$ can occur and in this case f is conjugate to the map $g(z) = z^2 - z$. This results can be strengthened as we see in section 4.

The following two theorems are very wonderful even though they only hold for one dimensional dynamical systems.

THEOREM 2.1. (Li-Yorke) *Suppose that f is a continuous real valued function defined on the closed interval J so that $F(J) \subseteq J$. If f has a period-3 point, then f has periodic points of all other periods.*

The *Sharkovsky ordering* of the positive integers is defined as follows;

$$\underbrace{3 \rhd 5 \rhd \cdots}_{\text{odd integers}} \rhd \underbrace{2 \cdot 3 \rhd 2 \cdot 5 \rhd \cdots}_{2\cdot(\text{odd integers})} \rhd \underbrace{2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \cdots}_{2^2\cdot(\text{odd integers})} \rhd \cdots \rhd \underbrace{\cdots \rhd 2^2 \rhd 2 \rhd 1}_{\text{powers of 2}}$$

THEOREM 2.2. (Sharkovsky) *Suppose that f is a continuous real valued function defined on the closed interval J so that $F(J) \subseteq J$. If f has a period-n point, then f has periodic points of period-m for all successors m of n in the Sharkovsky ordering.*

3. Construction of Φ

In this section, we devote to construct a function which has a period-n point but has no periodic points of all predecessors in the Sharkovsky ordering to give a proof of the following theorem A.

THEOREM A. *For each odd integer $n \geq 5$, there exists a continuous function f which has a period-n point but no period-$(n-2)$ point.*

For this, we mention the double of a function to guarantee the existence of a function which has a period-double point. For a given continuous function f defined on an interval I, the continuous function f_2, *the double of* f, is defined as follows;

Trisect I and $I \times I$ is decomposed into nine small squares. Compress the graph of f into the upper left corner of $I \times I$ and let X be the ending point of the graph of f. And the rest of the graph is filled with the two connected line segments, one is joining X with the left lower edge of the lower right corner of $I\times I$ and the other is the diagonal in the lower right corner of $I\times I$. Then, we have the following properties;

PROPOSITION 3.1. *The function $f_2(x)$, the double of a given function $f(x)$, has a periodic point of period-$2n$ at $\frac{x}{3}$ if and only if x is a period-n point for f.*

Proof. Without loss of generality, we may assume that I is the unit interval. So we have that for $x\in I$,

$$f_2(x)=\begin{cases}\frac{2}{3}+\frac{1}{3}f(x) & ,\ x\in[0,\frac{1}{3}]\\ -(2+f(\frac{1}{3}))(x-\frac{2}{3}) & ,\ x\in[\frac{1}{3},\frac{2}{3}]\\ x-\frac{2}{3} & ,\ x\in[\frac{2}{3},1]\end{cases}$$

and

$$f_2(\tfrac{x}{3})=\tfrac{2}{3}+\tfrac{1}{3}f(x).$$

Then, it is easy to check the results. □

PROPOSITION 3.2. *If $f_2(x)$, the double of $f(x)$, has a periodic point p that is not a fixed point, then either $3p$ or $3f_2(p)$ is a periodic point for f.*

Proof. Note that there is an integer n so that $f_2^{[n]}[\frac{1}{3},\frac{2}{3}]$ lies on $[0,\frac{1}{3}]$ or $[\frac{2}{3},1]$ since f_2 is linear on $[\frac{1}{3},\frac{2}{3}]$ with slope less than -2. Also since $f_2[0,\frac{1}{3}]\subset[\frac{2}{3},1]$ and $f_2[\frac{2}{3},1]\subset[0,\frac{1}{3}]$, there is no periodic point on $[\frac{1}{3},\frac{2}{3}]$. Suppose that p is a period-n point for f_2. Then we may assume that $p\in[0,\frac{1}{3}]$. Since

$$f(x)=3f_2(\tfrac{x}{3})-2 \quad\text{and}\quad f^{[n]}(x)=3f_2^{[2n]}(\tfrac{x}{3})\ ,\ n\geq 1$$

we have that $f(3p)=3f_2(p)-2$ and $f^{[n]}(3p)=3f_2^{[2n]}(p)=3p$. Thus $3p$ is a periodic point of f. □

In order to prove the above theorem A, we need a following lemma.

LEMMA 3.3. *Suppose that f is continuous on the closed interval J so that $f(J) \supseteq J$. Then f has a fixed point in J.*

Proof. Let r, s be minimum and maximum values of f in J, respectively and let $f(y) = r$ and $f(z) = s$ for some $y, z \in J$. Then since $r < y$, $z < s$, we have that if we let $g(x) = f(x) - x$, then

$$g(y) = f(y) - y \leq 0 \quad \text{and} \quad g(z) = f(z) - z \geq 0.$$

Thus, by the intermediate value theorem, there exists $x_0 \in J$ so that $g(x_0) = 0$. This means that x_0 is a fixed point of f. $\square$

From theorem A, we have the following result.

COLOLLARY 3.4. *For each odd integer $p \geq 5$, there exists a continuous function f so that f has a $2^m p$-periodic point but has no $2^m(p-2)$-periodic points for any positive integer m.*

Proof. By theorem A, it is immediate that for each odd integer $p \geq 5$, there exists a continuous function f which has a p-periodic point but no $(p-2)$-periodic points. Then by proposition 3.1 and 3.2, the double f_2 of f has a $2p$-periodic point but no $2(p-2)$-periodic points and also the double f_{2^2} of f_2 has a 2^2p-periodic point but no $2^2(p-2)$-periodic points. Inductively, we have that the double f_{2^m} of $f_{2^{m-1}}$ has a $2^m p$-periodic point but no $2^m(p-2)$-periodic points. $\square$

Now, we give the proof of theorem A.

Proof of Theorem A. For each positive integer m, define a continuous function which is piecewise linear joining the vertices given by

$$\Phi(k) = \begin{cases} m+2 & , \ k = 1 \\ 2m+5-k & , \ 2 \leq k \leq m+2 \\ 2m+4-k & , \ m+3 \leq k \leq 2m+3 \end{cases}$$

Then, Φ has a $(2m+3)$-cycle

$$\{1, m+2, m+3, m+1, m+4, m+5, m-1, \cdots, 3, 2m+2, 2, 2m+3\}$$

Now, consider the image of $[1,2]$ under the iterate map $\Phi^{[2m+1]}$ as following;

$$[1,2] \xrightarrow{\Phi} [m+2, 2m+3] \xrightarrow{\Phi} [1, m+3] \xrightarrow{\Phi} [m+1, 2m+3] \xrightarrow{\Phi}$$
$$[1, m+4] \xrightarrow{\Phi} [m, 2m+3] \xrightarrow{\Phi} [1, m+5] \xrightarrow{\Phi} [m-1, 2m+3] \xrightarrow{\Phi}$$
$$\cdots \xrightarrow{\Phi} [1, 2m+2] \xrightarrow{\Phi} [2, 2m+3]$$

That is $\Phi^{[2m+1]}([1,2]) = [2, 2m+3]$ and so there is no fixed point of $\Phi^{[2m+1]}$ on $[1,2]$. Similarly, we have that Φ has no period-$(2m+1)$ points in $[k, k+1]$ for all $1 \le k \le 2m+2$, $k \neq 2m+2$. For the interval $[m+2, m+3]$, we have

$$\Phi^{[2t]}[m+2, m+3] = [m+2-t, m+3+t]$$

and

$$\Phi^{[2t+1]}[m+2, m+3] = [m+1-t, m+3+t]$$

for $t = 1, 2, \cdots, m$. So, we have

$$\Phi^{[2m+1]}[m+2, m+3] \supset [m+2, m+3].$$

Thus $\Phi^{[2m+1]}$ has a fix point in $[m+2, m+3]$. But since Φ is decreasing on $\Phi^{[k]}[m+2, m+3]$ for $k = 0, 1, 2, \cdots, 2m$, the $(2m+1)$-iterate $\Phi^{[2m+1]}$ is decreasing on $[m+2, m+3]$. So, $\Phi^{[2m+1]}$ has only one fix point. But it is the fixed point of Φ since $\Phi[m+2, m+3] \subset [m+2, m+3]$. Hence there is no period-$(2m+1)$ points for Φ. $\square$

4. Fixed Points of Composite Map

In this section, for fixed points of composite map of some transcendental meromorphic functions, we devote to give a proof for the following theorem B.

THEOREM B. *Let f, g be transcendental meromorphic functions and f have at least three poles. Then the composite function $f \circ g$ has infinitely many fixed points.*

Before giving the proof, we make a mention of some remarkable known results for the fixed points of a transcendental function. It is clear that a transcendental entire function and a transcendental meromorphic function need not have fixed points by a simple example $f(z) = z + e^z$. But by the considering the map $\phi(z) = \frac{f(f(z))-z}{f(z)-z}$, each transcendental entire function f has at least one periodic point of period 2[11]. Later, It is generalized by Baker[2] and Rosenbloom[15] as follows ;

THEOREM 4.1. *A transcendental entire function has periodic points of all periods except at most one integer.*

THEOREM 4.2. [5,7] *Let f be transcendental meromorphic function which has exactly one pole and this pole is an omitted value. Then f has infinitely many periodic points of period $n \geq 2$.*

In fact, such map in theorem 4.2 need not have fixed points, as an example $f(z) = z + \frac{1}{g(z)}$, where $g(z)$ is a transcendental entire function.

Also Bergweiler[6] showed that the composite function $f \circ g$ of a transcendental entire function g and a transcendental meromorphic function f has infinitely many fixed points, and furthermore the same holds for the case of meromorphic function f which has at least two distinct poles.

Now, we end this note by giving the proof for Theorem B by using the following two Bergweiler's lemmas[6].

LEMMA 4.3. *Let f be a meromorphic function, and let z_0 be a pole of f of order p. Then there exists a function h, defined and analytic* *in a neighborhood of 0 such that* $h(0) = 0$ *and* $f(h(z) + z_0) = z^{-p}$ *for* $z \neq 0$.

LEMMA 4.4. *Let* f *and* g *be meromorphic functions. Then* $f \circ g$ *has infinitely many fixed points if and only if* $g \circ f$ *does.*

Proof of Theorem B. Let f be a transcendental meromorphic function with three distinct poles z_1, z_2 and z_3 of orders p_1, p_2 and p_3, respectively. Then by lemma 4.3, there exists functions h_j, defined and analytic in a neighborhood of 0 such that $h_j(0) = 0$ and $f(h_j(z) + z_j) = z^{-p_j}$, $z \neq 0$ for $j = 1, 2, 3$. Define

$$k_1(z) = h_2(z^{p_3 p_1} + z_2)$$
$$k_2(z) = h_3(z^{p_1 p_2} + z_3)$$
$$k_3(z) = h_1(z^{p_2 p_3} + z_1).$$

Then each k_j is analytic near 0 and

$$f(k_1(z)) = f(k_2(z)) = f(k_3(z)) = z^{-p_1 p_2 p_3}$$

in a punctured neighborhood of 0.

Now let u be defined by

$$u(z) = g(z^{-p_1 p_2 p_3}).$$

Then 0 is an essential singularity of u and we have

$$u(z) = g(f(k_1(z))) = g(f(k_2(z))) = g(f(k_3(z)))$$

in the punctured neighborhood of 0. Suppose that $f \circ g$ has only finitely many fixed points. Then by lemma 4.4, $g \circ f$ has at most finitely many fixed points. It follows that

$$u(z) \neq k_j(z)\ , \quad j = 1, 2, 3$$

in a punctured neighborhood of 0. Moreover, we have

$$k_i(z) \neq k_j(z) \quad \text{if} \quad i \neq j$$

in some neighborhood of 0 since $k_i(0) \neq k_j(0)$ for $i \neq j$. Now define

$$v(z) = \frac{u(z) - k_2(z)}{u(z) - k_3(z)} \cdot \frac{k_1(z) - k_3(z)}{k_1(z) - k_2(z)}.$$

Then v does not take the values 0, 1 and ∞ in some small punctured neighborhood of 0. Thus by Picard's theorem[1], v is constant, but it is a contradiction. $\square$

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Department of Information Security Analysis
Joongbu University
Kumsan 312-940, Korea

E-mail: sjyoo@joongbu.ac.kr