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THE n–DIMENSIONAL SP_α AND M_α–INTEGRALS

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Abstract. In this paper, we investigate the SP_α-integral and the M_α-integral defined on an interval of the n-dimensional Euclidean space $\mathbb{R}^n$. In particular, we show that these two integrals are equivalent.

In this paper, we introduce a Perron-type integral defined on an interval of the n-dimensional Euclidean space $\mathbb{R}^n$ using the strong α-major and α-minor functions. We shall call it the SP_α-integral. We also define a McShane-type integral(M_α-integral) and show that the SP_α-integral is equivalent to the M_α-integral.

For a subset E of the n-dimensional Euclidean space $\mathbb{R}^n$, the Lebesgue measure of E is denoted by $|E|$. For a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, the norm of x is $\|x\| = \max_{1 \le i \le n} |x_i|$ and the δ-neighborhood $U(x, \delta)$ of x is an open cube centered at x with sides equal to 2δ.

For an interval $I = [a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n]$ of $\mathbb{R}^n$ with $a_i < b_i$ for $i = 1, 2, \cdots, n$, we call the number $r(I) = \min_i(b_i - a_i)/\max_i(b_i - a_i)$ the *regularity* of I. If $r(I) > \alpha(\alpha \in (0, 1))$, then the interval I is said to be *α-regular*.

Throughout this paper, I_0 denotes a fixed interval in $\mathbb{R}^n$ and $\mathcal{I}$ the family of all subintervals of I_0. A positive function δ defined on a set $E \subset I_0$ is called a *gauge* on E. By $\mathcal{F}$ we denote the free full interval basis $\mathcal{F} = \{(I, x) : I \in \mathcal{I}, x \in I_0\}$. Note that for each pair $(I, x) \in \mathcal{F}$

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the point x need not belong to its interval I. For a given gauge δ and a given $\alpha \in (0,1)$, we write

$$\mathcal{F}^\alpha = \{(I,x) \in \mathcal{F} : r(I) > \alpha\},$$
$$\mathcal{F}^\alpha_\delta = \{(I,x) \in \mathcal{F} : r(I) > \alpha, I \subset U(x,\delta(x))\}.$$

For a set $E \subset I_0$, we write

$$\mathcal{F}(E) = \{(I,x) \in \mathcal{F} : I \subset E\},$$
$$\mathcal{F}[E] = \{(I,x) \in \mathcal{F} : x \in E\}.$$

A finite subset π of $\mathcal{F}$ is a *$\mathcal{F}$-partition* of I_0 if for distinct pairs (I_1,x_1) and (I_2,x_2) in π, I_1 and I_2 are nonoverlapping and $\cup_{(I,x)\in\pi} I = I_0$.

To define the SP_α-integral, we introduce first the definitions of the strong α-regular lower and upper derivates.

Definition 1. Let F be an interval function and let $x \in I_0$. The *strong α-regular lower and upper derivates* of F at x are defined by

$$\underline{SD}_\alpha F(x) = \sup_\delta \inf \left\{ \frac{F(I)}{|I|} : (I,x) \in \mathcal{F}^\alpha_\delta[\{x\}] \right\},$$
$$\overline{SD}_\alpha F(x) = \inf_\delta \sup \left\{ \frac{F(I)}{|I|} : (I,x) \in \mathcal{F}^\alpha_\delta[\{x\}] \right\}.$$

The function F is *strongly α-regularly differentiable at x* if

$$\underline{SD}_\alpha F(x) = \overline{SD}_\alpha F(x) \neq \pm\infty.$$

This common value is the *strong α-regular derivative* of F at x and is denoted by $SD_\alpha F(x)$.

It is easy to see that for any $0 < \alpha < \beta < 1$ and for any $x \in I_0$ we have

$$\underline{SD}_\alpha F(x) \leq \underline{SD}_\beta F(x) \leq \overline{SD}_\beta F(x) \leq \overline{SD}_\alpha F(x).$$

DEFINITION 2. Let f be a point function on I_0.

(a) An interval function M is a *strong α-major function* of f on I_0 if it is superadditive and $\underline{SD}_\alpha M(x) \geq f(x)$ for all $x \in I_0$.

(b) An interval function m is a *strong α-minor function* of f on I_0 if it is subadditive and $\overline{SD}_\alpha m(x) \leq f(x)$ for all $x \in I_0$.

DEFINITION 3. A function $f : I_0 \to \mathbb{R}$ is *SP_α-integrable* on I_0 if

$$-\infty < \sup\{m(I_0)\} = \inf\{M(I_0)\} < \infty,$$

where the supremum is taken over all strong α-minor functions of f and the infimum is taken over all strong α-major functions of f. This common value is the *SP_α-integral* of f on I_0 and is denoted by $(SP_\alpha)\int_{I_0} f$.

The following theorem is an immediate consequence of the definition.

THEOREM 4. *A function $f : I_0 \to \mathbb{R}$ is SP_α-integrable on I_0 if and only if for each $\epsilon > 0$ there exist a strong α-major function M and a strong α-minor function m on I_0 such that $M(I_0) - m(I_0) < \epsilon$.*

DEFINITION 5. Let $\alpha \in (0, 1)$. A function f on I_0 is *M_α-integrable* on I_0 with integral A if for each $\epsilon > 0$ there exists a gauge δ such that

$$\left| \sum_{(I,x)\in\pi} f(x)|I| - A \right| < \epsilon$$

for every $\mathcal{F}_\delta^\alpha$-partition π of I_0. We write $A = (M_\alpha)\int_{I_0} f$.

THEOREM 6. *Let $\alpha \in (0, 1)$. If a function f is SP_α-integrable on I_0, then f is M_α-integrable on I_0 and the integrals are equal.*

Proof. Suppose that f is SP_α-integrable on I_0 and let $\epsilon > 0$. Then there exist a strong α-major function M and a strong α-minor function m of f on I_0 such that

$$-\epsilon < m(I_0) - (SP_\alpha)\int_{I_0} f \leq 0 \leq M(I_0) - (SP_\alpha)\int_{I_0} f < \epsilon.$$

Since $\overline{SD}_\alpha m \le f \le \underline{SD}_\alpha M$ on I_0, for each $x \in I_0$ there exists $\delta(x) > 0$ such that

$$\frac{M(I)}{|I|} \ge f(x) - \epsilon \quad \text{and} \quad \frac{m(I)}{|I|} \le f(x) + \epsilon$$

whenever $(I, x) \in \mathcal{F}_\delta^\alpha[\{x\}]$.

If $\pi = \{(I_i, x_i) : 1 \le i \le n\}$ is any $\mathcal{F}_\delta^\alpha$-partition of I_0, then we have

$$\begin{aligned}
&\sum_{i=1}^n f(x_i)|I_i| - (SP_\alpha)\int_{I_0} f \\
&\quad \le \sum_{i=1}^n [f(x_i)|I_i| - M(I_i)] + M(I_0) - (SP_\alpha)\int_{I_0} f \\
&\quad < \epsilon \sum_{i=1}^n |I_i| + \epsilon = \epsilon(|I_0| + 1).
\end{aligned}$$

Similarly, using the minor function m

$$\sum_{i=1}^n f(x_i)|I_i| - (SP_\alpha)\int_{I_0} f > -\epsilon(|I_0| + 1).$$

It follows that f is M_α-integrable on I_0 and

$$(M_\alpha)\int_{I_0} f = (SP_\alpha)\int_{I_0} f.$$

□

THEOREM 7. *Let $\alpha \in (0,1)$. If f is M_α-integrable on I_0, then f is SP_α-integrable on I_0.*

Proof. Suppose that f is M_α-integrable on I_0 and let $\epsilon > 0$. Then there exists a gauge δ on I_0 such that

$$\left| \sum_{(I,x)\in\pi} f(x)|I| - (M_\alpha)\int_{I_0} f \right| < \epsilon$$

for every $\mathcal{F}^\alpha_\delta$-partition π of I_0. For each interval I, let

$$M(I) = \sup\Big\{ \sum_{(J,x)\in\pi} f(x)|J| : \pi \subset \mathcal{F}^\alpha_\delta(I)\Big\},$$

$$m(I) = \inf\Big\{ \sum_{(J,x)\in\pi} f(x)|J| : \pi \subset \mathcal{F}^\alpha_\delta(I)\Big\}.$$

Then it is easy to show that M is superadditive and m is subadditive. Fix a point $x \in I_0$. For each $(I,x) \in \mathcal{F}^\alpha_\delta[\{x\}]$, we have $M(I) \geq f(x)|I|$ and $\frac{M(I)}{|I|} \geq f(x)$. It follows that $\underline{SD}_\alpha M(x) \geq f(x)$. Similarly, $\overline{SD}_\alpha m(x) \leq f(x)$. Hence M is a strong α-major function of f on I_0 and m is a strong α-minor function of f on I_0.

Since

$$\Big| \sum_{(I,x)\in\pi_1} f(x)|I| - \sum_{(J,y)\in\pi_2} f(y)|J| \Big| < 2\epsilon$$

for any two $\mathcal{F}^\alpha_\delta$-partition π_1 and π_2 of I_0, we have $M(I_0) - m(I_0) \leq 2\epsilon$. By Theorem 4, f is SP_α-integrable on I_0. □

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