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# ON $\pi$-V-RINGS AND INTERMEDIATE NORMALIZING EXTENSIONS

Kang-Joo Min

Abstract. In this paper we study a ring over which every left module of finite length has an injective hull of finite length. We consider a ring that is a finite intermediate normalizing extension ring of such a ring. We also consider the subrings of such a ring.

Throughout this paper, all rings have identity and all modules are unital. Let $R$ be a ring. For a left or right $R$-module, $E(M)$ denote the injective hull of $M$. For an $R$-module $M$, $Le_R(M)$ denote the length of $M$[1]. Recall that a ring $R$ is a left V-ring if every simple left $R$-module is injective[4]. A left and right V-ring is called a V-ring. Rosenberg and Zelinsley[5] considered the rings over which every left module of finite length has an injective hull of finite length. Left V-rings form a special class in such rings. We will study such rings.

A ring $R$ is called a left(right) $\pi$-V-ring if for every simple left(right) $R$-module $M$, the injective hull $E(M)$ is of finite length. Let $n$ be a positive integer. A ring $R$ is called a left(right) $n$-V-ring if, for every simple left(right) $R$-module $M$, the length of $E(M)$ is less than or equal to $n$. Michler and Villamayor[4] proved that $R$ is a left V-ring if and only if every left $R$-module has the property that zero is an intersection of maximal submodules. We give a similar characterization for a left $\pi$-V-ring.

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THEOREM 1. [3] The following conditions are equivalent for a ring $R$:

(1) $R$ is a left $\pi$-V-ring.

(2) Every left $R$-module $M$ of finite length has an injective hull of finite length.

(3) For every left $R$-module $M$, the intersection of all submodules $N$ with $Le_R(M/N) < \infty$ is zero.

PROOF. (1) $\Rightarrow$ (2) Assume that $R$ is a left $\pi$-V-ring. Let $M$ be a left $R$-module of finite length. Then $M$ is Artinian and Noetherin. $U_1 \oplus U_2 \oplus \cdots \oplus U_k \leq_e M$ where $U_i$ is uniform. It suffices $E(U_i)$ is finite length. Let $A$ be a simple submodule of $U_i$. Then $E(A) = E(U_i)$ is of finite length.

(2) $\Rightarrow$ (1) Let $M$ be a simple module. Then $E(M)$ is of finite length.

(1) $\Rightarrow$ (3) Let $\Omega$ denote an irredundant set of representatives of the simple left $R$-modules. $C = \oplus_{T \in \Omega} E(T)$ is a cogenerator. Let $M$ be a nonzero left $R$-module. Then there exists an embedding $f : M \to \prod_{\alpha \in A} C_\alpha$ for some index set $A$ where $C_\alpha = C$. For $T \in \Omega$, let $P_{\alpha,T}$ be the projection from $\prod_{\alpha \in A} C_\alpha$ to the summand $E(T)$ of $C_\alpha$. Since $E(T)$ is of finite length by hypothesis, $M/\mathrm{Ker}(P_{\alpha,T} f)$ is of finite length. Since $\bigcap_{\alpha \in A, T \in \Omega} \mathrm{Ker}(P_{\alpha,T} f) = \mathrm{Ker}(f) = 0$, condition (3) is satisfied.

(3) $\Rightarrow$ (1) Let $T$ be a simple left $R$-module. By hypothesis, the intersection of submodules $N$ of the module $E(T)$ with $Le_R(E(T)/N) < \infty$ is zero. Hence there exists a submodule $U$ of $E(T)$ such that $E(T)/N$ is of finite length and $T \cap U = 0$. Since $E(T)$ is a essential extension of $T$, this implies $U = 0$. Hence $E(T)$ is of finite length. $\square$

A ring $S$ is called a finite normalizing extension of a ring $R$ if $R$ is a subring of $S$ and $S = \sum_{i=1}^{k} a_i R$ with $a_i R = R a_i$ for each $i$. A ring $S$ is called an excellent extension of $R$ if $S$ is a free normalizing extension of $R$ with a basis that includes 1 and $S$ is $R$-projective; that is, if $N$ is an $S$-submodule of $M_S$, the condition that $N_R$ is a direct summand of $M_R$ implies that $N_S$ is a direct summand of $M_S$.

Let $S$ be a finite normalizing extension of $R$. If $T$ is a subring of $S$ such that $R \subset T \subset S$, then $T$ is called an intermediate normalizing extension of $R$.

A short exact sequence $0 \to A \xrightarrow{\varphi} B \to C \to 0$ is the category of right $R$-modules is said to be pure(exact) if $0 \to A \bigotimes_R M \to B \bigotimes_R M \to C \bigotimes_R M \to 0$ is an exact sequence (of abelian groups) for any left $R$-module $M$. If this is the case, we say that $\varphi(A)$ is a pure submodule of $B$ (or that $B$ is a pure extension of $\varphi(A)$).

Theorem 2. Let $S$ be a finite normalizing extension of $R$ and $T$ be an intermediate normalizing extension of $R$ such that $T_T$ is a pure submodule of $S_T$. If $R$ is a left $\pi$-V-ring, then $T$ is a left $\pi$-V-ring.

Proof. By hypothesis, there is a finite set $\{a_1, a_2, \cdots, a_k\}$ of elements of $S$ such that $S = \sum_{i=1}^{k} a_i R$ and $a_i R = R a_i$ for each $i$. It is sufficient to show that for every left $T$-module $M$, the intersection of all $T$-submodules $N$ with $Le_R(M/N) < \infty$ is zero. Let $M$ be a nonzero left $T$-module and let $N$ be an $R$-submodule of $S \bigotimes_T M$ with $Le_R(S \bigotimes_T M)/N = m < \infty$ and $Le_R(S \bigotimes_T M/a_i^{-1}N) < \infty$ where $a_i^{-1}N = \{m \in S \bigotimes_T M \mid a_i m \in N\}$. Let $b(N) = \bigcap_{i=1}^{n} a_i^{-1}N$. $Le_R(S \bigotimes_T M/b(N)) < \infty$[3]. $b(N)$ is an $S$-submodule of $S \bigotimes_T M$ contained in $N$. $b(N)$ is a $T$-submodule of $M$[3]. $Le_R(M/b(N)) < \infty$. Since $R$ is a left $\pi$-V-ring, the intersection of $R$-submodules $N$ of $M$ with $Le_R(M/N) < \infty$ is zero. Therefore the intersection of $T$-

submodule $N'$ of $M$ with $Le_R(M/N') < \infty$ is zero. $\square$

THEOREM 3. Let $S$ be a finite normalizing extension of a ring $R$ and $R \subset T \subset S$ be an intermediate normalizing extension of $R$ such that $T_T$ is a pure submodule of $S_T$. If $S$ is a left $\pi$-V-ring, then $T$ is a left $\pi$-V-ring.

PROOF. By theorem 1.1, it is suffices to prove that for any left $T$-module $M$, the intersection of all $T$-submodule $N$ with $Le_T(M/N) < \infty$ is zero. Let $M$ be a nonzero $T$-module. By hypothesis, $M$ can be viewed as an $T$-submodule of $S \bigotimes_T M$. Let $L$ be an $S$-submodule of $S \bigotimes_T M$ with $Le_S(S \bigotimes_T M/L) < \infty$. By [2], $Le_T(S \bigotimes_T M/L) < \infty$. Hence $Le_T(M/M \cap L) < \infty$.

Since $S$ is a left $\pi$-V-ring, the intersection of $S$-submodules $L$ of $S \bigotimes_T M$ with $Le_T(S \bigotimes_T M/L) < \infty$ is zero. Therefore the intersection of $T$-submodules $N$ of $M$ with $Le_T(M/N) < \infty$ is zero. $\square$

COROLLARY 4. Let $S$ be a finite normalizing extension of a ring $R$ such that $R_R$ is a pure submodule of $S_R$. If $S$ is a left $\pi$-V-ring, then $R$ is a left $\pi$-V-ring.

A ring $S$ is a free normalizing extension of $R$ with a basis that includes 1; that is, there is a finite set $\{a_1, \cdots, a_n\} \subseteq S$ such that $a_1 = 1, S = Ra_1 + \cdots + Ra_n, a_iR = Ra_i$ for all $i = 1, 2, \cdots, n$ and $S$ is free with basis $\{a_1, a_2, \cdots, a_n\}$ as both a right and left $R$-module.

COROLLARY 5. If $S$ is a free normalizing extension of $R$, then $S$ is $\pi$-V-ring if and only if $R$ is a $\pi$-V-ring.

PROOF. Since $S$ is a free normalizing extension of $R$, $R_R$ is pure in $S_R$ and ${}_RR$ is pure in ${}_RS$. By corollary 4, $R$ is a left and right $\pi$-V-ring.

The converse follows from theorem 2. $\square$

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Department of Mathematics
Chungnam National University
Taejon 305-764, Korea

*E-mail*: kjmin@math.chungnam.ac.kr