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UNIQUE CONTINUATION FOR SCHRÖDINGER EQUATIONS

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Abstract. We prove a local unique continuation for Schrödinger equations with time independent coefficients. The method of proof combines a technique of Fourier-Gauss transformation and a Carleman inequality for parabolic operator.

1. Introduction. In this paper, we shall prove a local unique continuation result for Schrödinger equations with time independent coefficients. We consider the Schrödinger operator $L(x, \partial) = i\partial_t + P(x, \partial_x)$ on $\mathbb{R}^{n+1}$, where P is a positive elliptic second order operator with real valued coefficients. L is said to have the local unique continuation if u is a solution of $Lu = 0$ in a neighborhood of (0,0) and *supp* $u \subseteq \{(x,t) \in U : x_1 \geq 0\}$, where $U = \{(x,t) \in \mathbb{R}^{n+1} : x \in \Omega, t \in (-T,T)\}$, then $u = 0$ in a neighborhood of (0,0).

Concerning the unique continuation theorem, Rauch and Taylor [7] proved a sort of unique continuation theorem for hyperbolic equation with time independent coefficients. In order to prove this result they introduced a integral Fourier-Gauss type transformation. The first result in this direction are to be found in the work of Rauch and Taylor [7] and exploited by Lerner [4]. Using the same idea, we shall prove the main result. That is, our main tool will be the fundamental

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so-called Fourier- Gauss transformation and a Carleman inequality for parabolic operators.

This paper is organized as follows : In the second section we state our main results. The third section is devoted to prove a local unique continuation Theorem 2.1. Precisely, in Section 3.1 we state an elementary lemma of Fourier– Gauss transformation without proof. In Section 3.2 we make some preliminary and standard changes of variables in order to apply a Carleman inequality. In Section 3.3 we state a Carleman inequality for parabolic operator. In Section 3.4 we complete the proof of Theorem 2.1.

2. Statement of Main Result And Remarks

Let Ω be an open connected subset of $\mathbb{R}^n$ containing the origin. In this paper, we will use the following notation : $\bar{\Omega}$ is the closure of Ω, Ω^+ is the set $\{x \in \Omega : x_1 \geq 0\}$ and ∂_j means $\partial/\partial x_j$. We shall set $x = (x_1, x')$, with $x' = (x_2, \cdots, x_n)$ and $\xi = (\xi_1, \xi')$, $\xi' = (\xi_2, \cdots, \xi_n)$ the corresponding Fourier variable.

We consider now a Schrödinger operator :

$$L(x, \partial) = i\partial_t + P(x, \partial_x), \tag{2.1}$$

where

$$P(x, \partial_x) = \sum_{i,j=1}^{n} a_{i,j}(x)\partial_i\partial_j + \sum_{j=1}^{n} b_j(x)\partial_j + c(x) \tag{2.2}$$

is a positive elliptic second order differential operator with real valued leading coefficients in $C^1(\bar{\Omega})$ and depending on all the variables x and the other coefficients in $L^\infty(\Omega)$ and that they satisfying the ellipticity conditions :

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \alpha(x)|\xi|^2 \tag{2.3}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$ where $\alpha(x) > 0$ and $|\xi|^2 = \sum_{i=1}^{n} \xi_i^2$.

Now we can state our main theorem.

In the following theorem, we will denote by U for the set $\{(x,t) \in \mathbb{R}^{n+1} : x \in \Omega, -T < t < T\}$ for some constant T.

THEOREM 2.1. Let P be the operator defined in (2.2) and let (2.3) hold. If $u \in H^2_{loc}(U)$ is a solution of $Lu = 0$ in a neighborhood of the origin and *supp* $u \subseteq \{(x,t) \in U : x_1 \geq 0\}$, then u vanishes in a neighborhood of the origin.

Remark. The uniqueness of Theorem 2.1 is a local one. In Theorem 2.1, the hypothesis that the coefficients are independent of t is important. In fact, non-uniqueness examples can be found Lascar and Zuily [3]. That is, Lascar and Zuily [3] proved that there exists a smooth function $V(x,t)$ such that the Cauchy problem for the operator

$$\frac{1}{i}\partial_t - \triangle_x + V(x,t)$$

has not a local uniqueness property across the surface $\{x_1 = 0\}$ with the positive direction.

Remark. Kenig and Sogge [2] proved the unique continuation theorem for Schrödinger operator of the form $i\partial_t + \Delta_x$ on $\mathbb{R}^{n+1}$, if $n \geq 1$, and if $u(x,t)$ satisfies certain global integrability conditions as well as a differential inequality $|(i\partial_t + \Delta_x)u| \leq |Vu|$, where $V(x,t) \in L^{n+2/2}(\mathbb{R}^{n+1})$, then u vanishes identically if it vanishes in a halfspace.

Remark. In the case that the principal part of the operator P depends on all the x variables but P is elliptic, the result of hyperbolic operators has been proved by Robbiano [8] ; related results can be found Hörmander [1].

In the case that the real principal part of P depending only on one variable but P is elliptic, the result of weakly hyperbolic operators has been proved by Santo [9].

Remark. In the case that P is elliptic with smooth principal part, this uniqueness result has been proved by Lerner [4].

3. Proof of Theorem 2.1

The proof of Theorem 2.1 is based on Fourier-Gauss transformation and a Carleman inequality for parabolic operator.

3.1 Fourier-Gauss Transformation.

We define

$$\Lambda_{a,\lambda}(x,s) = \sqrt{\frac{\lambda}{2\pi}} \int_{-T_1}^{T_1} e^{-\frac{\lambda}{2}(is+a-t)^2} u(x,t)\ dt, \quad 0 < T_1 < T, \tag{3.1}$$

where λ is a large positive parameter and a a real number.

Assume that the set $\{x \in \mathbb{R}^n : |x| < r\}$ is contained in Ω and U' is the set $\{(x,s) \in \mathbb{R}^{n+1} : |x_i| < r/n for i = 1, \cdots, n, |s| < T_1/2\}$.

Let us state without proof an elementary lemma which we shall use in the sequel.

Lemma 3.1. If $|a| < T_1$, there exists a positive constant C_u depending on u, such that

(1) $\Lambda_{a,\lambda}(x,0) \longrightarrow u(x,a)\ in\ L^2\ as\ \lambda \to +\infty$,
(2) $||\Lambda_{a,\lambda}||_{H^1(U')} \leq C_u \lambda^{\frac{1}{2}} e^{\frac{\lambda {T_1}^2}{8}}$,
(3) (3) $\Lambda_{a,\lambda}(x,s) = 0$ if $(x,s) \in U'$ and $x_1 \leq 0$.

Let

$$L = i\partial_t + P(x, \partial_x) \tag{3.2}$$

be an operator satisfying (2.2) and (2.3).

The following lemma can easily verified to be an analogue to Lemma 2 in Robbiano [8].

Lemma 3.2. If $|a| < T_1$, there exist a positive constant C_u depending on u, such that

$$\|\tilde{L}\Lambda_{a,\lambda}\|_{L^2(U')} \le C_u \lambda^{\frac{1}{2}} e^{\frac{\lambda T_1{}^2}{8} - \frac{\lambda}{2}(T_1 - |a|)^2}, \tag{3.3}$$

where $\tilde{L} = -\partial_s + P(x, \partial_x)$.

Section 3.2-3.3 are standard and follows very closely the proof [11].

3.2 Preliminary Transformation.

We consider the Holmgren transformation :

$$\begin{cases} y_1 = x_1 + (|x'|^2 + t^2), \\ y' = x', \\ s = t. \end{cases} \tag{3.4}$$

By this change of variables, we will deduce

$$\tilde{\tilde{L}}(x, \partial) = -\partial_s + a(x,s)(\partial x_1 + A(x, s, \partial'_x))^2 + B(x, s, \partial'_x) + \tilde{b}_i(x)\partial_i + \tilde{c}(x). \tag{3.5}$$

where A and B are order 1 and 2, respectively. Note that $a(x,s) \neq 0$ in a neighborhood of the origin since the hyperplane $x_1 = 0$ is not characteristic.

The equation

$$\frac{\partial \theta}{\partial x_1} + A(x, s, \partial'_x)\theta = 0 \tag{3.6}$$

has $n-1$-independent solutions $\theta_2, \cdots, \theta_n$ which satisfy :

$$\theta_j(0, x', s) = x_j, \quad j = 2, \cdots, n. \tag{3.7}$$

Now, the change of variables $(x_1, x_2, \cdots, x_n, s) \to (x_1, \theta_2, \cdots, \theta_n, s)$ satisfies the required properties.

Dividing by the coefficient of $\partial^2_{x_1}$ (in the new variables), the operator L can finally be written as

(3.8)
$$Q = -\frac{1}{\tilde{a}(x,s)}\partial_s + \partial^2_{x_1} + \frac{1}{\tilde{a}(x,s)}R(x,s,\partial' x) + \frac{1}{\tilde{a}(x,s)}\sum_i \tilde{\tilde{b}}_i(x)\partial_i + \frac{1}{\tilde{a}(x,s)}\tilde{\tilde{c}}(x)$$

where $\tilde{a}$ is C^1, R is an operator of order ≤ 2, with C^1 coefficients ; the coefficients $\tilde{\tilde{b}}_i$, $\tilde{\tilde{c}}$ obtained from b_i, c in (2.2) satisfying the smoothness hypothesis of Theorem 2.1.

3.3 Carleman Inequality.

There are many versions of Carleman inequality for parabolic operator (Nirenberg [6], Mizohata [5], Saut and Scheurer [11]). Here, especially, we shall apply lemma 1.5 of Saut and Scheurer [11] to Q defined by (3.8).

LEMMA 3.3. (see [11], Lemma 1.5) Under the hypothesis of Theorem 2.1 on the coefficients of Q, there exist positive constants δ_0', K, M' such that for $0 < \delta < \delta_0$ and $\tau\delta > M'$,

(3.9)
$$||e^{\tau\psi}Qv||^2_{L^2} \geq K\{\tau^3\delta^2||e^{\tau\psi}v||^2_{L^2} + \tau||e^{\tau\psi}\partial x_1 v||^2_{L^2} + \tau\delta||e^{\tau\psi}\partial'_x v||^2_{L^2}\},$$

for all $v \in C_0^\infty(u)$ with sufficiently small support and where ψ is defined by

$$\psi(x,s) = (x_1 - \delta)^2 + \delta^2(|x'|^2 + s^2). \tag{3.10}$$

3.4 End of the Proof of Theorem 2.1.

Since

$$supp\tilde{\tilde{\Lambda}}_{a,\lambda} \subset \{(y,s) : y_1 \geq |y'|^2 + s^2 \geq \epsilon(|\tilde{y}'|^2 + \tilde{s}^2)\},$$

we have

$$\begin{aligned}\{(y,s); y_1 \geq |y'|^2 + s^2\} &\subset \{(y,s); (y-\delta)^2 + \delta^2(|y'|^2 + s^2) \leq \delta^2\} \\ &\equiv \{(y,s); \psi(y,s) \leq \psi(0,0)\}.\end{aligned} \tag{3.11}$$

Now we let $\chi \in C_0^\infty(U')$ be a smooth function such that $\chi \equiv 1$ in a neighborhood $\tilde{U}$ of the origin.

We set $\omega_{a,\lambda} = \chi\tilde{\tilde{\Lambda}}_{a,\lambda}$; from (3.9),

$$||e^{\tau\psi} Q\omega_{a,\lambda}||_{L^2(U'\cap\tilde{\tilde{\Omega}}^+\times I)} \geq K\tau^{3/2}\delta\ ||e^{\tau\psi}\tilde{\tilde{\Lambda}}_{a,\lambda}||_{L^2(U'\cap\tilde{\tilde{\Omega}}^+\times I)}, \tag{3.12}$$

where $I = (-T_1, T_1)$ and ψ is defined by (3.10).

On the other hand,

$$Q\omega_{a,\lambda} = \chi Q\tilde{\tilde{\Lambda}}_{a,\lambda} + [Q,\chi]\tilde{\tilde{\Lambda}}_{a,\lambda},$$

where $[Q,\chi]$ is a first order operator which support is contained in $(U' \setminus \tilde{U}) \cap \tilde{\tilde{\Omega}}^+ \times I$ and the commutator of two operators A and B is defined as the operator $[A,B]v = A(Bv) - B(Av)$.

Since *supp* $Q\omega_{a,\lambda} \subset$ *supp* $\omega_{a,\lambda}$, there exist positive constants k_1 and k_2 with $k_1 > k_2$ such that

$$\{(U'\setminus\tilde{U})\cap\tilde{\tilde{\Omega}}^+ \times I\} \subset \{(y,s); \psi(y,s) \leq \psi(0,0) - k_1 = \delta^2 - k_1\}, \tag{3.13}$$

and let

$$(\tilde{U}/N \cap \tilde{\tilde{\Omega}}^+ \times I) \subset (U' \cap \Omega^+ \times I)$$

be a neighborhood of (0,0) and for N large enough such that

$$(3.14)\quad (\tilde{U}/N \cap \tilde{\tilde{\Omega}}^{+} \times I) \subset \{(y,s); \psi(y,s) > \psi(0,0) - k_2 = \delta^2 - k_2\}.$$

So that by (2) of Lemma 3.1 and (3.13), we obtain

$$(3.15)\qquad \|e^{\tau\psi}[Q,\chi]\tilde{\tilde{\Lambda}}_{a,\lambda}\|_{L^2\{(U'\setminus\tilde{U})\cap\tilde{\tilde{\Omega}}^{+}\times I\}} \leq e^{\tau(\delta^2-k_1)}\tilde{\tilde{C}}_u\lambda^{\frac{1}{2}}e^{\frac{\lambda {T_1}^2}{8}},$$

and by (3) of Lemma 3.1, we have

$$(3.16)\qquad \|e^{\tau\psi}\chi Q\tilde{\tilde{\Lambda}}_{a,\lambda}\|_{L^2(U'\cap\tilde{\tilde{\Omega}}^{+}\times I)} \leq e^{\tau\delta^2}\tilde{\tilde{C}}_u\lambda^{\frac{1}{2}}e^{\frac{\lambda {T_1}^2}{8}-\frac{\lambda}{2}(T_1-|a|)^2}.$$

From (3.14), the inequality (3.12) becomes,

$$(3.17)\quad \|e^{\tau\psi}Q\omega_{a,\lambda}\|_{L^2(U'\cap\tilde{\tilde{\Omega}}^{+}\times I)} \geq k\tau^{3/2}\delta e^{\tau(\delta^2-k_2)}\|\tilde{\tilde{\Lambda}}_{a,\lambda}\|_{L^2(\frac{\tilde{U}}{N}\cap\tilde{\tilde{\Omega}}^{+}\times I)}.$$

Now we use inequality (3.15) and (3.16) combined with (3.17) and we set $\tau = \nu\lambda$, where ν will be chosen later on, then we have

$$(3.18)\quad \|\tilde{\tilde{\Lambda}}_{a,\lambda}\|_{L^2(\frac{\tilde{U}}{N}\cap\tilde{\tilde{\Omega}}^{+}\times I)} \leq (\tilde{\tilde{C}}_u/K\cdot\delta)\nu^{-3/2}\frac{1}{\lambda}\left[e^{\lambda\{-\nu(k_1-k_2)+\frac{T_1^2}{8}\}} + +e^{\lambda\{\nu k_2+\frac{T_1^2}{8}-\frac{1}{2}(T_1-|a|)^2\}}\right].$$

We want to show that $||\tilde{\tilde{\Lambda}}_{a,\lambda}||_{L^2(\frac{\tilde{U}}{N}\cap\tilde{\tilde{\Omega}}^{+}\times I)}$ tends to 0 when λ tends to $+\infty$. For this purpose, we have to prepare the followings.

From (3.18), we have

$$\begin{cases} -\nu(k_1 - k_2) + {T_1}^2/8 < 0, \\ \nu k_2 + {T_1}^2/8 - (T_1 - |a|)^2/2 < 0. \end{cases} \tag{3.19}$$

Then we will find ν satisfying (3.19) if

$$k_2 {T_1}^2/8(k_1 - k_2) + {T_1}^2/8 < (T_1 - |a|)^2/2. \tag{3.20}$$

since $k_2 < 1/N$ and $0 \leq a \leq T_1/10$.

Then we get

$$\lim_{\lambda \to +\infty} \|\tilde{\tilde{\Lambda}}_{a,\lambda}\|_{L^2(\frac{\tilde{U}}{N} \cap \tilde{\tilde{\Omega}}^+ \times I)} = 0.$$

Hence u is zero by (1) of Lemma 3.1. The proof of Theorem 2.1 is complete.

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