JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 15, No. 2, December 2002

FREE PROBABILITY THEORY AND ITS APPLICATION

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Abstract. We prove a simplicity of the C^*-algebra generated by some C^*-subalgebra and a Haar unitary in a free product of finite von Neumann algebras. Some examples and questions are given.

1. Introduction and Preliminaries

The study of the free probability theory has seen rapid and impressive progress since it was introduced by Voiculescu in the framework of operator algebras. This free probability theory has turned out to be very powerful in the study of von Neumann algebras associated with free products of discrete groups, in particular, of free group factors.

In the non-commutative probability theory there is a notion of the free product of finite von Neumann algebras with specified traces, for which one has $L(G_1) * L(G_2) \simeq L(G_1 * G_2)$ where $L(G)$ is the finite von Neumann algebra associated to a discrete group G. In particular, the reduced free product of C^*-algebras is closely related to the free product of groups via the reduced group C^*-algebra. The notion of freeness in operator algebras can be viewed as an abstract extension of freeness in groups.

There are many questions about (reduced) free products. One of most basic questions concerns simplicity of free product C^*-algebras

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supported by Chungnam National University fund in 2001.

Received by the editors on December 6, 2002 .

2000 *Mathematics Subject Classifications*: Primary 46L09, 46L54, Secondary 46L05, 46L10.

Key words and phrases: Reduced free product, Haar unitary, simplicity, property T.

or C^*-subalgebras of free product von Neumann algebras. In 1975, Powers [9] showed that the reduced group C^*-algebra $C^*_r(G)$ is simple and has a unique trace when G is the free group F_2 on two generators. This example of the simple C^*-algebra intrigued the further deep structure theory for C^*-algebras and notably K-theory for C^*-algebras. Furthermore, his method turned out to be a prototype for inferring the simplicity of C^*-algebras. See the survey [3] for a detail discussion, some properties of operator algebras associated to the free groups and related "geometric" groups giving rise to algebras with similar properties.

Avitzour [1] introduced a free product of C^*-algebras with faithful states and generalized Powers' result to free products of C^*-algebras. A necessary and sufficient condition for the simplicity of the reduced free product of finite-dimensional abelian C^*-algebras is given by Dykema [4]. In Voiculescu's free probability theory, basic objects inherits the asymptotic properties of families of random matrices and are isomorphic to the free group factors. Furthermore, Voiculescu introduced the definition of the free entropy of an n-tuple of self-adjoint elements in a tracial W^*-probability space. See [10] and its references for some new development on free group factors.

In this paper, we concern with the simplicity and primality of (reduced) free products. First, we show the simplicity of the C^*-algebra generated by some C^*-subalgebra and a Haar unitary in a free product of finite von Neumann algebras and the uniqueness of a trace. This result has already been proved by Dykema [4], but we are showing by the different method extending directly the Powers' technique. We give some example and questions concerning primality, especially, free group factors in the last section.

The remainder of this section is concerned with definitions of free products of von Neumann algebras and reduced free products of C^*-algebras. Let $\mathcal{A}$ be a unital algebra with a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(1) = 1$. A family $(\mathcal{A}_i)_{i\in I}$ of unital subalgebras in $\mathcal{A}$ is called *free* with respect to ϕ if

$$\phi(a_1 a_2 \cdots a_n) = 0$$

whenever $a_j \in \mathcal{A}_{i_j}$, $i_j \neq i_{j+1}$ and $\phi(a_j) = 0$ for all j. This freeness is conceptually analogous to independence in the classical probability theory, though completely noncommutative. See the monograph [10] for a good introduction, more details on free products and various applications related to free group factors.

We first recall some basic facts about free products of finite von Neumann algebras. Suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are finite von Neumann algebras with a faithful normal normalized states ϕ_1 and ϕ_2, respectively.

From the GNS construction, we can get representations $(\mathcal{H}_1, \pi_1, \xi_1)$ and $(\mathcal{H}_2, \pi_2, \xi_2)$. We can identify $\pi_i(\mathfrak{M}_i)$ with $\mathfrak{M}_i$ $(i = 1, 2)$ because of the faithfulness and normality of states. We will also identify ξ with ξ_1 and ξ_2 which we assume to be unit vectors.

Then the free product of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ (with respect to a distinguished unit vector ξ) is the Hilbert space $\mathcal{H}$ given by

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \left(\bigoplus_{(i_1\neq i_2\neq\cdots\neq i_n)} \mathcal{H}^\circ_{i_1} \otimes \mathcal{H}^\circ_{i_2} \otimes \cdots \mathcal{H}^\circ_{i_1} \right),$$

where $\mathcal{H}^\circ_{i_j}$ is the orthogonal complement of ξ_{i_j} in $\mathcal{H}_{i_j}$ for $i_j = 1, 2$. Note that $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ act naturally on $\mathcal{H}$ from the left when we identify $\mathcal{H}$ with $\mathcal{H}_i \otimes \mathcal{K}_i$ where $\mathcal{K}_i$ are those tensors in $\mathcal{H}$ not beginning in $\mathcal{H}^\circ_i$ $(i = 1, 2)$. Especially, we have the embeddings $\sigma_i : \mathfrak{M}_i \hookrightarrow \mathcal{B}(\mathcal{H}_i)$ $(i = 1, 2)$. Then the *free product* $(\mathfrak{M}_1 * \mathfrak{M}_2, \phi)$ of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ (with respect to ϕ_1 and ϕ_2) is the von Neumann algebra on $\mathcal{H}$ generated by $\sigma_1(\mathfrak{M}_1)$ and $\sigma_2(\mathfrak{M}_2)$ where ϕ is the vector state induced by ξ, that is, $\phi = \phi_1 * \phi_2 = \langle \cdot \xi, \xi \rangle$. $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are free with respect to ϕ in the following sense for $x_j \in \mathfrak{M}_1$ and $y_j \in \mathfrak{M}_2$, $\phi(x_1 y_1 \cdots x_n y_n) = 0$ when $\phi(x_j) = \phi(y_j) = 0$ for all j.

The reduced free product of C^*-algebras can be constructed as follows [14]: Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital C^*-algebras with states ϕ_1 and ϕ_2 whose associated GNS representations are faithful. The *reduced* free product of $(\mathcal{A}_1, \phi_1)$ and $(\mathcal{A}_2, \phi_2)$ is the unital C^*-algebra $(\mathcal{A}, \phi)$ with unital embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}$ such that

(i) the GNS representation associated with ϕ is faithful on $\mathcal{A}$;
(ii) $\phi|_{\mathcal{A}_i} = \phi_i$;
(iii) $\mathcal{A}_1$ and $\mathcal{A}_2$ are free with respect to ϕ;
(iv) $\mathcal{A}$ is generated by $\mathcal{A}_1$ and $\mathcal{A}_2$.

We introduced the definition of the reduced free product algebra. There is also full free product algebra. However, we mainly deal with the reduced free product algebra.

2. Simplicity of C^*-subalgebras in a von Neumann algebra free product

Throughout this section, $\mathfrak{M}_j$ denotes a finite von Neumann algebra with a faithful finite normal trace τ_j $(j = 1, 2)$, unless specified otherwise. The algebraic free product $\mathfrak{M}_1 *_a \mathfrak{M}_2$ of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is embedded into $\mathfrak{M}_1 * \mathfrak{M}_2$. We identify $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with $\pi_1(\mathfrak{M}_1)$ and $\pi_2(\mathfrak{M}_2)$, respectively. Then all finite linear combinations of the identity element I and elements of the form $x_1 x_2 \cdots x_n$ $(n \geq 1)$ form an ultraweakly dense $*$-subalgebra of $\mathfrak{M}_1 * \mathfrak{M}_2$ where $x_j \in \mathfrak{M}_{i_j}$ and $\tau_{i_j}(x_j) = 0$ with $i_j = 1, 2$ and $i_j \neq i_{j+1}$.

Given a finite von Neumann algebra $(\mathfrak{M}, \tau)$ with a trace, a *Haar* *unitary* (with respect to a trace τ) is a unitary, $u \in \mathfrak{M}$, such that $\tau(u^n) = 0$ for every non-zero integer n. This is equivalent to τ of the spectral measure of u being Haar measure on the unit circle.

Let $\mathcal{A}$ be a unital norm separable C^*-subalgebra of $\mathfrak{M}_1$ which is ultraweakly dense in $\mathfrak{M}_1$ and contains a Haar unitary u. Let v be a Haar unitary in $\mathfrak{M}_2$ and $\mathfrak{A}$ the C^*-subalgebra in $\mathfrak{M}_1 * \mathfrak{M}_2$ generated by $\mathcal{A}$ and v. We denote still by τ the restriction of the trace on $\mathfrak{M}_1 * \mathfrak{M}_2$ to $\mathfrak{A}$. We define sets $\mathcal{F}_i$ $(i = 1, 2)$ as follows:

$\mathcal{F}_1 = \{v^{n_0} a_1 v^{n_1} \cdots a_k v^{n_k} : a_j \in \mathcal{A}$ with $\tau_1(a_j) = 0, k \geq 1$ and $n_j \neq 0 (1 \leq j \leq k-1)\}$, $\mathcal{F}_2 = \{v^n : n \in \mathbb{Z} \setminus \{0\}\}$.

Note that n_0 and n_k may be zero. Let $\mathcal{F}_0 = \mathcal{F}_1 \cup \mathcal{F}_2$ and $\mathcal{F} = \mathcal{F}_0 \cup \{I\}$.

All finite linear combinations of elements in $\mathcal{F}$, denoted by $\mathbb{C}\mathcal{F}$, form a norm dense $*$-subalgebra of $\mathfrak{A}$. The length of an element in $\mathcal{F}$ is defined as follows:

the length of $v^{n_0} a_1 v^{n_1} \cdots a_k v^{n_k}$ is $|n_0| + \cdots + |n_k|$ and the length of I is 0.

The following lemma is well-known, so that we will omit the proof.

LEMMA 2.1. *Let $x_1, \cdots, x_k$ be a finite sequence of operators on a Hilbert space such that image subspaces $\mathrm{Im}(x_1), \cdots, \mathrm{Im}(x_k)$ are pairwise orthogonal. Then we have*

$$\|x_1 + \cdots + x_k\| \leq \sqrt{k} \max_{1 \leq j \leq k} \|x_j\|.$$

The following proposition which extends Powers' result to a free product of finite von Neumann algebras are the same as that of Dykema [4], but we will give a proof by a direct variation on Powers' proof for the reduced C^*-algebra of non-abelian free groups.

THEOREM 2.2. *$\mathfrak{A}$ is simple and has a unique trace.*

Proof. Suppose that $\mathfrak{J}$ is a non-zero two-sided ideal in $\mathfrak{A}$. We can choose a non-zero positive element x in $\mathfrak{J}$ with $\tau(x) \neq 0$. Multiplying x by some constant $\lambda \in \mathbb{C}$, we may assume that $\tau(x) = 1$. From the Kaplansky's density theorem, we know that for any $\epsilon > 0$ there is a self-adjoint element y in $\mathbb{C}\mathcal{F}$ such that $\|x - y\| < \epsilon$ and $\|y\| \leq \|x\|$. Moreover, we may assume that $\tau(y) = 1$. Thus we can write $y = I + \sum_{j=1}^{n} \lambda_j y_j$ where $\lambda_j \in \mathbb{C}$ and $y_j \in \mathcal{F}_0$ for $j = 1, \cdots, n$ since traces of all elements in $\mathcal{F}_0$ are zero.

Let $m_0 - 1$ be the maximal length of $y_1, \cdots, y_n$, that is,

$$m_0 - 1 = \max\{|y_j| : 1 \leq j \leq n\}.$$

Then the elements $v^{-m_0} y_j v^{m_0}$ begin and end with a nonzero power of v for all $j = 1, \cdots, n$. Let $\mathcal{C}$ be the set of elements in $\mathcal{F}_0$ which begins by v^{m_0} (followed by a non-trivial element in $\mathcal{A}$, or by nothing at all) and set $\mathcal{D} = \mathcal{F} - \mathcal{C}$. Then we see that $\mathcal{C}$ and $\mathcal{D}$ are orthogonal. From above argument, we also see that $y_j\mathcal{C}$ and $\mathcal{C}$ are orthogonal for $j = 1, \cdots, n$. Since $\mathcal{D} = \mathcal{F} - \mathcal{C}$, one can see that $u^i v^{-m_0}\mathcal{D}$ and $u^j v^{-m_0}\mathcal{D}$ are orthogonal whenever $i \neq j$. We know that $L^2(\mathfrak{A}, \tau) = L^2(\mathfrak{M}_1 * \{v\}'', \tau)$ since $\mathcal{A}$ is ultraweakly dense in $\mathfrak{M}_1$.

Let $\mathcal{K}$ be the closed subspace spanned by $\mathcal{D}$ and P the projection from $L^2(\mathfrak{A}, \tau)$ onto $\mathcal{K}$. For each positive integer j, we set $w_j = u^j v^{-m_0}$ and $Q_j = w_j P w_j^{-1}$. Since $w_i\mathcal{D}$ and $w_j\mathcal{D}$ are orthogonal for $i \neq j$, we see that $Q_1, Q_2, \cdots$ are pairwise orthogonal projections. $y_j\mathcal{C}$ and $\mathcal{C}$ are also orthogonal, so that one has

$$(I - P) y_j (I - P) = 0 \quad \text{for } j = 1, 2, \cdots, n.$$

From the definition of each Q_j, we get $I - P = w_j^{-1}(I - Q_j) w_j$. Since $(I - P) y_j (I - P) = 0$ for each $1 \leq j \leq n$, we have $(I -$

$P)(\sum_{j=1}^{n}\lambda_j y_j)(I-P)=0$, so that

$$0=(I-Q_j)w_j\big(\sum_{j=1}^{n}\lambda_j y_j\big)w_j^{-1}(I-Q_j).$$

Now we show that for each integer $k\geq 1$, one has

$$\left\|\frac{1}{k}\sum_{j=1}^{k}w_j\big(\sum_{i=1}^{n}\lambda_i y_i\big)w_j^{-1}\right\|\leq\frac{2}{\sqrt{k}}\left\|\sum_{i=1}^{n}\lambda_i y_i\right\|.$$

For any unit vector $\xi\in L^2(\mathfrak{A},\tau)$, we have that

$$\begin{aligned}
\left|\left\langle\frac{1}{k}\sum_{j=1}^{k}w_j\big(\sum_{i=1}^{n}\lambda_i y_i\big)w_j^{-1}\xi,\xi\right\rangle\right| &\leq\frac{1}{k}\sum_{j=1}^{k}\left|\left\langle w_j\big(\sum_{i=1}^{n}\lambda_i y_i\big)w_j^{-1}\xi,\xi\right\rangle\right|\\
&=\frac{1}{k}\sum_{j=1}^{k}\left|\left\langle(I-Q_j+Q_j)w_j\big(\sum_{i=1}^{n}\lambda_i y_i\big)w_j^{-1}(I-Q_j+Q_j)\xi,\xi\right\rangle\right|\\
&\leq\frac{1}{k}\sum_{j=1}^{k}\left\{\left|\left\langle(I-Q_j)w_j\big(\sum_{i=1}^{n}\lambda_i y_i\big)w_j^{-1}(I-Q_j)\xi,\xi\right\rangle\right|\right.\\
&\quad+\left|\left\langle(I-Q_j)w_j\big(\sum_{i=1}^{n}\lambda_i y_i\big)w_j^{-1}Q_j\xi,\xi\right\rangle\right|\\
&\quad\left.+\left|\left\langle Q_j w_j\big(\sum_{i=1}^{n}\lambda_i y_i\big)w_j^{-1}\xi,\xi\right\rangle\right|\right\}\\
&\leq\frac{1}{k}\sum_{j=1}^{k}\left\{\left\|(I-Q_j)w_j\big(\sum_{i=1}^{n}\lambda_i y_i\big)w_j^{-1}\right\|\|Q_j\xi\|\right.\\
&\quad\left.+\left\|w_j\big(\sum_{i=1}^{n}\lambda_i y_i\big)w_j^{-1}\right\|\|Q_j\xi\|\right\}\\
&\leq\frac{1}{k}\sum_{j=1}^{k}\left\|\sum_{i=1}^{n}\lambda_i y_i\right\|(\|Q_j\xi\|+\|Q_j\xi\|)\\
&\leq\frac{2}{\sqrt{k}}\left\|\sum_{i=1}^{n}\lambda_i y_i\right\|.
\end{aligned}$$

Since y is a self-adjoint element, $y - I = \sum_{i=1}^{n} \lambda_i y_i$ is also self-adjoint. Hence we have the inequality

$$\left\| \frac{1}{k} \sum_{j=1}^{k} w_j \Big(\sum_{i=1}^{n} \lambda_i y_i \Big) w_j^{-1} \right\| \leq \frac{2}{\sqrt{k}} \left\| \sum_{i=1}^{n} \lambda_i y_i \right\|.$$

Next we prove that

$$\left\| I - \frac{1}{k} \sum_{j=1}^{k} w_j x w_j^{-1} \right\| \leq \frac{2}{\sqrt{k}} \|x\| + \epsilon$$

for each integer $k \geq 1$. Using the triangle inequality, we have

$$\begin{aligned}
\left\| I - \frac{1}{k} \sum_{j=1}^{k} w_j x w_j^{-1} \right\| &\leq \left\| I - \frac{1}{k} \sum_{j=1}^{k} w_j y w_j^{-1} \right\| \\
&\quad + \left\| \frac{1}{k} \sum_{j=1}^{k} w_j y w_j^{-1} - \frac{1}{k} \sum_{j=1}^{k} w_j x w_j^{-1} \right\| \\
&= \left\| \frac{1}{k} \sum_{j=1}^{k} w_j \Big(\sum_{i=1}^{n} \lambda_i y_i \Big) w_j^{-1} \right\| + \|y - x\| \\
&\leq \frac{2}{\sqrt{k}} \left\| \sum_{i=1}^{n} \lambda_i y_i \right\| + \epsilon \\
&\leq \frac{2}{\sqrt{k}} \|x\| + \epsilon.
\end{aligned}$$

For sufficiently large k, we get the inequality

$$\left\| I - \frac{1}{k} \sum_{j=1}^{k} w_j x w_j^{-1} \right\| < 1.$$

It follows that $b = \frac{1}{k} \sum_{j=1}^{k} w_j x w_j^{-1}$ is invertible. Since $x \in \mathfrak{J}$ and $\mathfrak{J}$ is a two-sided ideal, b lies in $\mathfrak{J}$. This implies that the identity element I lies in $\mathfrak{J}$, so that $\mathfrak{J} = \mathfrak{A}$. Therefore, $\mathfrak{A}$ is simple.

To show that $\mathfrak{A}$ has a unique trace, let τ' be any normalized trace on $\mathfrak{A}$. Suppose that $x \in \mathfrak{A}$ and $\epsilon > 0$ is given. Then it follows from above proof that there are unitaries $w_j \in \mathcal{F}$ and $\lambda_j \in \mathbb{C}$ $(j = 1, \cdots, n)$ with $\sum_j \lambda_j = 1$ such that

$$\left\| \tau(x)I - \sum_{j=1}^{n} \lambda_j w_j x w_j^* \right\| < \epsilon.$$

Hence, we have

$$\begin{aligned} |\tau'(\tau(x)I - \sum_{j=1}^{n} \lambda_j w_j x w_j^*)| &= |\tau(x) - \sum_{j=1}^{n} \lambda_j \tau'(w_j x w_j^*)| \\ &= |\tau(x) - \tau'(x)| < \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\tau(x) = \tau'(x)$ for all $x \in \mathfrak{A}$. Hence $\mathfrak{A}$ has a unique trace. □

3 Some remarks and questions

Ge [6] asked a question suggested by the primality of free group factors: *Is the relative commutant of a non-atomic injective (or abelian) von Neumann subalgebra of $L(\mathbb{F}_n)$ in $L(\mathbb{F}_n)$ always injective?* We can ask the similar question for free product II_1-factors, that is, *for any non-atomic injective subalgebra $\mathcal{B}$ of a free product II_1-factor $\mathcal{M}$, is the relative commutant of $\mathcal{B}$ in $\mathcal{M}$ always injective?* Here we would like to answer this question in the negative. A counterexample is directly obtained from Theorem 5.2 in [5]. For the convenience of a reader, we will give the example.

EXAMPLE 3.1. Suppose that the group G is the direct product of an i.c.c. amenable countable discrete group H with a countable discrete group G_1. Then $L(H)$ is the hyperfinite II_1-factor and we denote it by $\mathcal{R}_2$ which is generated by two unitary generators U_2, V_2 with the relation $U_2V_2 = e^{2\pi i\theta'}V_2U_2$. Let $\mathcal{M}_1 = \mathcal{R}_1 * L(G)$ where $\mathcal{R}_1$ is generated by unitary generators U_1, V_1 with $U_1V_1 = e^{2\pi i\theta}V_1U_1$. If we choose θ' such that $2\theta' = \theta$, then the mapping α given by $\alpha(U_1) = U_2$ and $\alpha(V_1) = V_2^2$ determines an isomorphism of $\mathcal{R}_1$ into $\mathcal{R}_2$. Let $\mathcal{N} = M_2(\mathbb{C}) \otimes \mathcal{M}_1$ and let $\mathcal{R}$ be the subalgebra of $\mathcal{N}$ consisting of all elements $\begin{pmatrix} x & 0 \\ 0 & \alpha(x) \end{pmatrix}$ for $x \in \mathcal{R}_1$. We denote by $\mathcal{M}$ the free product of $\mathcal{N}$ and $\mathcal{N}_1$ where $\mathcal{N}_1$ is any type II_1-factor. By Corollary 4.2, we have that $\mathcal{R}' \cap \mathcal{M}$ is contained in $\mathcal{N}$, so that $\mathcal{R}' \cap \mathcal{M} = \mathcal{R}' \cap \mathcal{N}$. From Theorem 5.2 in [5], we obtain that the relative commutant of $\mathcal{R}$ in $\mathcal{M}$ is not injective if G_1 is not amenable.

We showed that if G is a discrete i.c.c group with property T of Kazhdan, $L(G)$ is not isomorphic to $\mathcal{N} \otimes L(\mathbb{F}_2)$ for some factor $\mathcal{N}$ of type II_1 [7]. We ask if the factor with property T is prime. Although we have seen that free products (representing "freeness") are quite different from tensor products (representing "independence"), we strongly suspect that all free product II_1-factors are prime. It is proved in [6] that $L(\mathbb{F}_2)$ is prime using Voiculescu's free entropy theory, but we don't know *if* $L(\mathbb{F}_2) \otimes L(\mathbb{F}_2)$ *is* $*$*-isomorphic to* $L(\mathbb{F}_2) \otimes L(\mathbb{F}_2) \otimes L(\mathbb{F}_2)$. Hence we ask the following question:

Question 3.2. *Is* $\mathrm{Aut}(L(\mathbb{F}_2) \otimes L(\mathbb{F}_2))$ *(resp.,* $\mathrm{Out}(L(\mathbb{F}_2) \otimes L(\mathbb{F}_2))\mathbb{F}$*) isomorphic to* $\mathrm{Aut}(L(\mathbb{F}_2) \otimes L(\mathbb{F}_2) \otimes L(\mathbb{F}_2))$ *(resp.,* $\mathrm{Out}(L(\mathbb{F}_2) \otimes L(\mathbb{F}_2) \otimes L(\mathbb{F}_2))$*)?*

In connection with Question 3.2, it may make sense to ask *if* $\mathrm{Aut}(L(\mathbb{F}_2))$ *(resp.,* $\mathrm{Out}(L(\mathbb{F}_2))$*) is isomorphic to* $\mathrm{Aut}(L(\mathbb{F}_2) \otimes L(\mathbb{F}_2))$ *(resp.,* $\mathrm{Out}(L(\mathbb{F}_2) \otimes L(\mathbb{F}_2))$*).*

Remark 3.3 Kadison first noticed that the interchange of two free generators of $\mathbb{F}_2$ induces an outer automorphism of $L(\mathbb{F}_2)$. Hence permutations on the n generators induce automorphisms of $L(\mathbb{F}_n)$, which give rise to actions of the permutation group S_n on $L(\mathbb{F}_n)$. Since each generator of the free group corresponds to a unitary operator that generates a non-atomic abelian subalgebra of $L(\mathbb{F}_n)$, from Lemma 3.5 in [7] we obtain that this action is outer in the sense that S_n embeds into $\mathrm{Out}(L(\mathbb{F}_n))$ faithfully. Automorphisms and their outer conjugacy classes of the factor $L(\mathbb{F}_\infty)$ have been studied by J. Phillips. We hope that further studies on automorphisms of free group factors $L(\mathbb{F}_n)$ may lead us to understand more connections among free group factors $L(\mathbb{F}_n)$ for $n = 2, 3, \dots, \infty$.

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