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QUATERNIONIC HEISENBERG GROUP

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Abstract. We shall study the automorphism group of the quaternionic Heisenberg group $\mathcal{H}_7(\mathbb{H}) = \mathbb{R}^3\tilde{\times}\mathbb{H}$ which is important to investigate an *almost Bieberbach group* of a 7-dimensinal infra-nilmanifold and show that $\mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) \cong \mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3) \rtimes O(J;2,2)$.

1. Introduction

The complex Heisenberg group $\mathcal{H}_{2n+1}(\mathbb{C}) = \mathbb{R}\tilde{\times}\mathbb{C}^n$ with group operation given by

$$(s,z)(t,z') = (s+t+2\mathrm{Im}\{z\bar{z}'\}, z+z')$$

for $z = (z_1, z_2, \cdots, z_n), z' = (z'_1, z'_2, \cdots, z'_n) \in \mathbb{C}^n$, where Im $\{z\bar{z}'\}$ is the imaginary part of the complex number $z_1\bar{z}'_1 + z_2\bar{z}'_2 + \cdots + z_n\bar{z}'_n$ is a simply connected 2-step nilpotent Lie group with the center $\mathcal{Z}(\mathcal{H}_{2n+1}(\mathbb{C})) = \mathbb{R}$. Let M be an infra-nilmanifold with $\mathcal{H}_{2n+1}(\mathbb{C})$-geometry. It is well known (see for example [4]) that an almost Bieberbach group contains a cocompact lattice of $\mathcal{H}_{2n+1}(\mathbb{C})$ with index bounded above by a universal constant I. That is, I is the maximal order of the holonomy groups. It is shown in [7] that when $n = 2, I = 24$.

For the case of the quaternionic Heisenberg group $\mathcal{H}_{4n+3}(\mathbb{H}) = \mathbb{R}^3\tilde{\times}\mathbb{H}^n$ with group operation given by

$$(s,q)(t,q') = (s+t+2\mathrm{Im}\{q\bar{q}'\}, q+q')$$

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for $q=(q_1,q_2,\cdots,q_n), q'=(q'_1,q'_2,\cdots,q'_n)\in\mathbb{H}^n$, where Im $\{q\bar{q}'\}$ is the imaginary part of the quaternion number $q_1\bar{q}'_1+q_2\bar{q}'_2+\cdots+q_n\bar{q}'_n$ seen as an element of $\mathbb{R}^3$, it is shown in [3] that when $n=1, I_2=48$. The number I_2 is significant in its own right. According to a recent work (see [5, Corollary 5.3]), it is related to the minimum volume of a quaternionic hyperbolic orbifolds.

For $q=x_1+ix_2+jx_3+kx_4\in\mathbb{H}$, write

$$q(1)=x_1,\ q(i)=x_2,\ q(j)=x_3,\ q(k)=x_4.$$

Also we define

$$\bar{q}=x_1-ix_2-jx_3-kx_4$$
$$\breve{q}=-x_1+ix_2-jx_3-kx_4.$$

For $q,q'\in\mathbb{H}$, we define $q\odot q'$ as follows:

$$q\odot q'=q\breve{q}'(1)+q\breve{q}'(i)+q\bar{q}'(j)+q\bar{q}'(k).$$

The main concern of this paper is the quaternionic Heisenberg group $\mathcal{H}_{4n+3}(\mathbb{H})=\mathbb{R}^3\tilde{\times}\mathbb{H}^n$ with group operation given by

$$(s,q)(t,q')=(s+t+2\mathrm{Im}\{q\odot q'\},q+q')$$

for $q=(q_1,q_2,\cdots,q_n), q'=(q'_1,q'_2,\cdots,q'_n)\in\mathbb{H}^n$, where Im $\{q\odot q'\}$ is the imaginary part of the quaternion number $q_1\odot q'_1+q_2\odot q'_2+\cdots+q_n\odot q'_n$ seen as an element of $\mathbb{R}^3$. Then $\mathcal{H}_{4n+3}(\mathbb{H})$ is a simply connected 2-step nilpotent Lie group with the center $\mathcal{Z}(\mathcal{H}_{4n+3}(\mathbb{H}))=\mathbb{R}^3$.

Let M be an infra-nilmanifold with $\mathcal{H}_{4n+3}(\mathbb{H})$-geometry; that is, $M=\Pi\backslash\mathcal{H}_{4n+3}(\mathbb{H})$, where $\Pi\subset\mathcal{H}_{4n+3}(\mathbb{H})\rtimes C$ is a torsion free, discrete subgroup with compact quotient, where C is a compact subgroup of $\mathrm{Aut}(\mathcal{H}_{4n+3}(\mathbb{H}))$. Such a group Π is called an *almost Bieberbach group* (= AB-group). Since

$$\Pi\cap\mathcal{Z}(\mathcal{H}_{4n+3}(\mathbb{H}))\cong\mathbb{Z}^3$$

is a lattice of $\mathcal{Z}(\mathcal{H}_{4n+3}(\mathbb{H}))$, M fits

$$T^3 \to M \to N,$$

a Seifert 3-torus "bundle" over a $4n$-dimensional flat orbitfold. When there is no singular point, it is a genuine bundle over the base space N which is a flat Riemannian $4n$-manifold.

In this paper, we shall study the automorphism group of $\mathcal{H}_7(\mathbb{H}) = \mathbb{R}^3 \tilde{\times} \mathbb{H}$ and show that $\mathrm{Aut}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) \cong \mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes O(J;2,2)$.

2. The Quaternionic Heisenberg Group

From now on, we shall use $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$ rather than $\mathbb{R}^3 \tilde{\times} \mathbb{H}$. We identify $\mathbb{R}^3 \tilde{\times} \mathbb{H}$ with $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$ by

$$(s, q = x_1 + ix_2 + jx_3 + kx_4) \longleftrightarrow (s, x = [x_1, x_2, x_3, x_4]^t).$$

Accordingly, we introduce a new notation for Im $\{q \odot q'\}$. For

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

define

$$\begin{aligned}
&\mathcal{I}(x,y) \\
&= \left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} - \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix}, - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} - \begin{vmatrix} x_4 & y_4 \\ x_2 & y_2 \end{vmatrix}, - \begin{vmatrix} x_1 & y_1 \\ x_4 & y_4 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \right)^t \\
&= (x^t J_1 y, x^t J_2 y, x^t J_3 y)^t,
\end{aligned}$$

where $()^t$ denotes the transpose of a matrix, and

$$J_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$J_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly $\mathcal{I}(x,y) = -\mathcal{I}(y,x)$ and $\mathcal{I}(x,y)$ corresponds to Im $\{q \odot q'\}$. Thus the group operation in $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$ becomes

$$(s,x)(t,y) = (s+t+2\mathcal{I}(x,y),\ x+y).$$

Since $\mathcal{I}(x,\pm x) = 0$, we see easily that

$$(s,x)^{-1} = (-s,-x).$$

Thus we have

$$[(s,x),\ (t,y)] = (s,x)^{-1}(t,y)^{-1}(s,x)(t,y) = (4\mathcal{I}(x,y),\ 0).$$

Therefore the center of $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$, $\mathcal{Z}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4)$ is $\mathbb{R}^3$. Thus we have shown the following lemma.

LEMMA 2.1. *$\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$ is a simply connected 2-step nilpotent Lie group with the center $\mathcal{Z}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) = \mathbb{R}^3$.* □

Let

$$\sigma = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$K_i = \sigma^{-1} J_i \sigma,\ i = 1,2,3.$$

Then $J_1,\ J_2,\ J_3$ together with $K_1,\ K_2,\ K_3$ form a linear basis for the vector space $\mathfrak{so}(4)$ of the skew-symmetric matrices. Therefore,

$$\mathfrak{so}(4) = < J_1, J_2, J_3 > \oplus < K_1, K_2, K_3 >$$

as vector spaces. Notice that each subspace fail to be a Lie subalgebra.

In order to understand the automorphism group of $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$, we need to study more general setting: For any $C \in \mathrm{GL}(4,\mathbb{R})$ and $V \in \mathfrak{so}(4)$,

$$J_C(V) = C^t V C$$

defines a linear isomorphism $J_C : \mathfrak{so}(4) \to \mathfrak{so}(4)$. In fact, with respect to the basis $\{J_1, J_2, J_3, K_1, K_2, K_3\}$, it turns out that $\det(J_C) = (\det(C))^3$.

Define

$$O(J;2,2) = \{C \in \mathrm{GL}(4,\mathbb{R}) \mid C^t J_i C \in < J_1, J_2, J_3 >\}.$$

That is, $C \in O(J;2,2)$ if and only if the map J_C leaves the subspace spanned by J_1, J_2, J_3 invariant. Therefore,

$$C^t J_i C = \lambda_{i1} J_1 + \lambda_{i2} J_2 + \lambda_{i3} J_3, \quad \lambda_{ij} \in \mathbb{R},$$

for $i = 1, 2, 3$. It turns out then, the matrix $\lambda = (\lambda_{ij})$ is non-singular.

Now we form the column vector

$$J = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix}$$

with entries the matrices J_1, J_2, J_3. With some abuse of notation, we can write

$$O(J;2,2) = \{C \in \mathrm{GL}(4,\mathbb{R}) \mid C^t J C = \lambda J,\ \lambda \in \mathrm{GL}(3,\mathbb{R})\}.$$

Clearly $C \in O(J;2,2)$ is a closed subgroup of $\mathrm{GL}(4,\mathbb{R})$. For $C \in O(J;2,2)$, let $\widehat{C} \in \mathrm{GL}(3,\mathbb{R})$ denote the nonsingular 3×3 matrix λ which satisfies $C^t J C = \lambda J$. So,

$$C^t J C = \widehat{C} J.$$

Then a map $C \to \widehat{C}$ defines a homomorphism $\widehat{\ } : O(J;2,2) \to \mathrm{GL}(3,\mathbb{R})$.

Note that as a map $J_C : \mathfrak{so}(4) \to \mathfrak{so}(4)$, with respect to the ordered basis $\{J_1, J_2, J_3, K_1, K_2, K_3\}$, J_C is of the form

$$J_C = \begin{bmatrix} \widehat{C} & 0 \\ 0 & * \end{bmatrix}.$$

Since the center, $\mathcal{Z}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) = \mathbb{R}^3$, is a characteristic subgroup of $\mathbb{R}^3\tilde{\times}\mathbb{R}^4$, every automorphism of $\mathbb{R}^3\tilde{\times}\mathbb{R}^4$ restricts to an automorphism of $\mathbb{R}^3$. Consequently an automorphism of $\mathbb{R}^3\tilde{\times}\mathbb{R}^4$ induces an automorphism on the quotient group $\mathbb{R}^4$. Thus there is a natural homomorphism $\mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) \to \mathrm{Aut}(\mathbb{R}^3) \times \mathrm{Aut}(\mathbb{R}^4)$ defined by $\theta \mapsto (\hat{\theta}, \bar{\theta})$.

LEMMA 2.2. $\mathrm{Im}\,\{\ \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) \to \mathrm{Aut}(\mathbb{R}^4)\} = O(J;2,2)$. *Moreover, the exact sequence* $\mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) \to O(J;2,2) \to 1$ *splits.*

Proof. Let $\theta \in \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)$. Then

$$(\hat{\theta}, \bar{\theta}) \in \mathrm{Aut}(\mathbb{R}^3) \times \mathrm{Aut}(\mathbb{R}^4).$$

Since $[(s,x),(t,y)] = (4\mathcal{I}(x,y),\ 0)$,

$$\begin{aligned}\theta[(s,x),(t,y)] &= \theta(4\mathcal{I}(x,y),\ 0) = (\hat{\theta}(4\mathcal{I}(x,y)),\ \bar{\theta}(0)\ ) \\ &= (4\hat{\theta}(\mathcal{I}(x,y)),\ 0)\end{aligned}$$

and

$$[\theta(s,x),\ \theta(t,y)] = [(*,\ \bar{\theta}(x)),\ (*,\ \bar{\theta}(y)] = (4\mathcal{I}(\bar{\theta}(x),\ \bar{\theta}(y)),\ 0)$$

yield

$$\mathcal{I}(\bar{\theta}(x),\ \bar{\theta}(y)) = \hat{\theta}(\mathcal{I}(x,y)),$$

or, equivalently,

$$(\bar{\theta}(x)^t J_1 \bar{\theta}(y),\ \bar{\theta}(x)^t J_2 \bar{\theta}(y),\ \bar{\theta}(x)^t J_3 \bar{\theta}(y))^t\ = \hat{\theta} \cdot (x^t J_1 y, x^t J_2 y, x^t J_3 y)^t$$

for all x, y. This happens if and only if $\bar{\theta}^t J \bar{\theta} = \hat{\theta} J$. Therefore, $\bar{\theta} \in O(J;2,2)$.

Conversely, suppose that $\bar{\theta} \in O(J;2,2)$, i.e., $\bar{\theta}^t J \bar{\theta} = \lambda J$ is satisfied for some $\lambda \in \mathrm{GL}(3,\mathbb{R})$. We define $\theta : \mathbb{R}^3\tilde{\times}\mathbb{R}^4 \to \mathbb{R}^3\tilde{\times}\mathbb{R}^4$ by

$$\theta(s,x) = (\lambda \cdot s,\ \bar{\theta}(x)).$$

Then

$$\begin{aligned}
\theta((s,x)\cdot(t,y)) &= \theta(s+t+2\mathcal{I}(x,y),\ x+y)\\
&= (\lambda\cdot(s+t+2\mathcal{I}(x,y)),\ \bar\theta(x+y))\\
&= (\lambda\cdot s+\lambda\cdot t+\lambda\cdot 2\mathcal{I}(x,y),\bar\theta(x+y)),\\
\theta(s,x)\cdot\theta(t,y) &= (\lambda\cdot s,\bar\theta(x))\cdot(\lambda\cdot t,\bar\theta(y))\\
&= (\lambda\cdot s+\lambda\cdot t+2\mathcal{I}(\bar\theta(x),\bar\theta(y)),\ \bar\theta(x)+\bar\theta(y)).
\end{aligned}$$

Now the condition $\bar\theta^t J\bar\theta=\lambda J$ guarantees that

$$\lambda\cdot\mathcal{I}(x,y)=\mathcal{I}(\bar\theta(x),\bar\theta(y)).$$

Thus θ is an automorphism of $\mathbb{R}^3\tilde\times\mathbb{R}^4$. Moreover, this defines a split homomorphism $O(J;2,2)\to\mathrm{Aut}(\mathbb{R}^3\tilde\times\mathbb{R}^4)$. $\square$

PROPOSITION 2.3. (Structure of Aut $(\mathbb{R}^3\tilde\times\mathbb{R}^4)$)

$$\mathrm{Aut}(\mathbb{R}^3\tilde\times\mathbb{R}^4)\cong\mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3)\rtimes O(J;2,2)$$

where an element $(\eta,A)\in\mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3)\rtimes O(J;2,2)$ *acts by*

$$(\eta,A)(s,x)=(\hat{A}s+\eta(x),Ax).$$

Proof. Let $\theta\in\mathrm{Aut}(\mathbb{R}^3\tilde\times\mathbb{R}^4)$. Then we have the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3\tilde\times\mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & 1\\
 & & \big\downarrow{\scriptstyle\hat\theta} & & \big\downarrow{\scriptstyle\theta} & & \big\downarrow{\scriptstyle\bar\theta} & & \\
1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3\tilde\times\mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & 1
\end{array}$$

Thus

$$\theta(s,x)=(\hat\theta(s)+\eta(s,x),\bar\theta(x))$$

for $(s,x) \in \mathbb{R}^3 \tilde{\times} \mathbb{R}^4$, where $\eta : \mathbb{R}^3 \tilde{\times} \mathbb{R}^4 \to \mathbb{R}^3$. Since θ is a homomorphism, one can show that η is a homomorphism, i.e.,

$$\eta((s,x)(t,y)) = \eta(s,x) + \eta(t,y).$$

In particular, set $x = 0$.

$$(\hat{\theta}(s), 0) = \theta(s,0) = (\hat{\theta}(s) + \eta(s,0), 0)$$

implies that $\eta(s,0) = 0$ for all $s \in \mathbb{R}^3$, and thus

$$\eta(s,x) = \eta((s,0)(0,x)) = \eta(s,0) + \eta(0,x) = \eta(0,x).$$

Hence $\eta \in \mathrm{Hom}\ (\mathbb{R}^4, \mathbb{R}^3)$.

Let us find out the kernel of the surjective homomorphism of Lemma 2.2:

$$\mathrm{Aut}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) \to O(J;2,2),\ \ \theta \mapsto \bar{\theta}.$$

Suppose that $\theta \in \mathrm{Aut}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4)$ with $\bar{\theta} = \mathrm{id}_{\mathbb{R}^4}$. Then $\hat{\theta} = \mathrm{id}_{\mathbb{R}^3}$ and thus

$$\theta(s,x) = (s + \eta(x), x)$$

for some $\eta \in \mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3)$.

Conversely given $\eta \in \mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3)$, define $\theta \in \mathrm{Aut}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4)$ by $\theta(s,x) = (s + \eta(x), x)$. Clearly this θ lies in the kernel of the homomorphism. Hence we have a short exact sequence

$$1 \to \mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3) \to \mathrm{Aut}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) \to O(J;2,2) \to 1.$$

By Lemma 2.2, this sequence splits. $\square$

Note that $\mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes O(J;2,2)$ is sitting inside

$$\mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes (\mathrm{GL}(3,\mathbb{R}) \times O(J;2,2))$$

as $(\eta, (\hat{A}, A))$, and the action of $O(J;2,2)$ on $\mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3)$ is

$$^{A}\eta(x) = \hat{A} \cdot \eta(A^{-1}x).$$

The group operation on $(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) \rtimes O(J;2,2)$ is given by

$$\begin{aligned} &((s,x),A)((t,y),B) = ((s,x) \cdot^{A} (t,y),\ AB) \\ &= ((s,x) \cdot (\hat{A}t, Ay),\ AB) \\ &= ((s + \hat{A}t + 2\mathcal{I}(x, Ay), x + Ay),\ AB). \end{aligned}$$

3. Application

Let $\Pi \subset \mathbb{R}^3 \tilde{\times} \mathbb{R}^4 \rtimes \mathrm{Aut}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4)$ be an AB-group. Then it is well known that $\Gamma = \Pi \cap (\mathbb{R}^3 \tilde{\times} \mathbb{R}^4)$, the pure translations in Π, is the maximal normal nilpotent subgroup, and $\Phi = \Pi/\Gamma$, the holonomy group of Π, is finite. Since Γ is a lattice of $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$, $\Gamma \cap \mathcal{Z}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4)$ is a lattice of $\mathcal{Z}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) = \mathbb{R}^3$, and $\Gamma/\Gamma \cap \mathcal{Z}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4)$ is a lattice of $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4 / \mathcal{Z}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) = \mathbb{R}^4$. Thus

$$\Gamma \cap \mathcal{Z}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) \cong \mathbb{Z}^3$$

and

$$\Gamma/\Gamma \cap \mathcal{Z}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) \cong \mathbb{Z}^4.$$

Consider the following natural commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{R}^3 & \xrightarrow[=]{} & \mathbb{R}^3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbb{R}^3\tilde{\times}\mathbb{R}^4 & \longrightarrow & \mathbb{R}^3\tilde{\times}\mathbb{R}^4 \rtimes \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) & \longrightarrow & \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \rtimes \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) & \longrightarrow & \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

Recall from Proposition 2.3 that an element

$$(\eta, A) \in \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) = \mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes O(J; 2, 2)$$

acts on $(s, x) \in \mathbb{R}^3\tilde{\times}\mathbb{R}^4$ by

$$(\eta, A)(s, x) = (\hat{A}s + \eta(x), Ax).$$

Thus $O(J; 2, 2)$ acts on $\mathbb{R}^3$ via the homomorphism

$$\widehat{\ }: O(J; 2, 2) \to \mathrm{GL}(3, \mathbb{R}),$$

and $O(J; 2, 2)$ acts on $\mathbb{R}^4$ by matrix multiplication $O(J; 2, 2) \times \mathbb{R}^4 \to \mathbb{R}^4$.

Let $Q = \Pi/\mathbb{Z}^3$. Then the above diagram induces the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}^3 & \xrightarrow[=]{} & \mathbb{Z}^3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & Q & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

Here $\Phi \subset O(J;2,2)$ acts on $\mathbb{Z}^4$ by matrix multiplication, and on $\mathbb{Z}^3$ via the homomorphism $\hat{}: O(J;2,2) \to \mathrm{GL}(3,\mathbb{R})$.

Recall from [6, Proposition 2] that a virtually free abelian group

$$1 \to \mathbb{Z}^4 \to Q \to \Phi \to 1$$

is a crystallographic group if and only if the centralizer of $\mathbb{Z}^4$ in Q has no torsion elements. Since Φ acts effectively on $\mathbb{Z}^4$, it follows that Q is naturally a 4-dimensional crystallographic group.

Let Π be an AB-group for $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$. Then Π is a torsion free extension of $\mathbb{Z}^3$ by a 4-dimensional crystallographic group Q so that

$$1 \to \mathbb{Z}^3 \to \Pi \to Q \to 1$$

is exact.

Construction from Q. For each 4-dimensional crystallographic group Q, we shall check if there exists a construction from Q; that is, a torsion free $\Pi \subset \mathcal{H}_7(\mathbb{H}) \rtimes \mathrm{Aut}(\mathcal{H}_7(\mathbb{H}))$ fitting the short exact sequence

$$1 \to \mathbb{Z}^3 \to \Pi \to Q \to 1.$$

This is the key notion for our arguments and construction. We have a complete classification of 4-dimensional crystallographic groups (Q's in the above statement). There are 4783 4-dimensional crystallographic groups up to isomorphism. We shall use the presentations of the 4-dimensional crystallographic groups given in the book [1]:

H. Brown, R. Bülow, J. Neubüser, H. Wondratschek and H. Zassenhaus, *Crystallographic Groups of Four-Dimensional Spaces*, John Wiley Sons, New York, 1978.

The crystallographic groups will be called Q, and every Q has an explicit representation $Q \to \mathbb{R}^4 \rtimes \mathrm{GL}(4, \mathbb{Z})$ in this book.

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