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ON THE RELATIONSHIP BETWEEN STABLE DOMAINS AND CRITICAL ORBITS

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ABSTRACT. This paper is concerned with some properties of stable domains and limit functions. Using the relationship between cycles of periodic stable domains and orbits of critical points and using the Sullivan theorem [19], we prove that the value of a constant limit function in some stable domain for a rational function f of degree at least two lies in the closure of the set of critical orbits of f.

1. Introduction

The complex sphere is divided into two sets, namely, the Fatou set and the Julia set, by the iteration of a rational function (see [4, 5, 6, 9, 11, 12, 13, 18, 19]). Components, called stable domains, of the Fatou set are classified to many different types by considering the limit functions of sequences of iterates.

One of the remarkable results about the stable domains is the Sullivan's no wandering domains theorem [19]. He used the theory of quasiconformal mappings to solve one of the most important problems for any rational functions which were remained open after the work of Fatou [11] and Julia [13]. And Shishikura [17] showed that a rational function of degree d has at most $2(d-1)$ distinct cycles of periodic domains.

According to the above classification, a stable domain which has a constant limit function contains a fixed point in its closure [4]. Also

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in the dynamics of a rational function f, there is some relationship between cycles of periodic stable domains and orbits of critical points of f.

In this note, we devote to prove, by using the Sullivan's no wandering domains theorem, the value of a constant limit function in some stable domain for a rational function f of degree at most two lies in the closure of the forward images of the set of critical points of f.

2. Preliminaries

For a rational map f defined on the complex sphere $\overline{C}$, we denote the *degree* of f by $\deg(f)$, the maximum of $\deg(P)$ and $\deg(Q)$, where $f(z) = \frac{P(z)}{Q(z)}$, and $P(z), Q(z)$ are relatively prime polynomials. Also we denote f^n the n-th *iterate* of f, that is, $f^0(z) = z$ and $f^n(z) = f^{n-1}(f(z))$ for all $z \in \overline{C}$ and all $n \in \mathbb{N}$. For $z_0 \in \overline{C}$, the set $\{f^n(z_0) \mid n \geq 0\}$ is called the *orbit* of z_0 under the function f.

The dynamics of a rational function f induces a subdivision of the complex sphere $\overline{C}$ into two sets, namely, the *Fatou* set and the *Julia* set. The *Fatou* set $F(f)$ of a rational function f is defined to be the maximal open subset of $\overline{C}$ on which $\{f^n \mid n \geq 0\}$ is equicontinuous, and the *Julia* set $J(f)$ of f is its complement in $\overline{C}$. The components of $F(f)$ are called the *stable domains*.

Since equicontinuity is closely related to normal families of analytic functions, we can use the more powerful results about normal families, namely, Arzelà-Ascoli Theorem, Vitali Theorem and Montel's Normality Criterion [1, 4, 8, 9, 18], to derive further information about the Fatou and Julia sets. In fact, the set of normal points of f is the Fatou set of f and so we have the property that f and f^n have identical Fatou sets and Julia sets. In particular, for a non-linear polynomial f, $f^n \to \infty$ on some neighborhood of ∞. So by the above mentioned theorems we have the following [4];

PROPOSITION 2.1. *Let f be a polynomial of degree at least two. Then $f^n \to \infty$ on the stable domain Ω_∞ which contains ∞.*

Note that a domain which has at least three boundary points is called a *hyperbolic domain*. By using Montel's Normality Criterion, the following is immediate.

PROPOSITION 2.2. *Any invariant hyperbolic domain is a part of the Fatou set.*

Let f be a map from a set X into itself. A subset U of X is said to be *forward*, *backward* or *completely invariant* if $f(U) \subset U$, $f^{-1}(U) \subset U$ or $f(U) = U = f^{-1}(U)$, respectively.

PROPOSITION 2.3. *A completely invariant and closed subset K of $\overline{C}$ containing at least three points is contained in the Julia set.*

Proof. Suppose that a closed subset K of $\overline{C}$ is completely invariant. Then $\Omega = \overline{C} - K$ is completely invariant. Hence Ω is open and for each $n \in \mathbb{N}$, $f^n : \Omega \to \Omega$ is an open map. If we choose three distinct elements $a, b, c \in K$, then $f^n(\Omega) \cap \{a, b, c\} = \phi$ for all $n \in \mathbb{N}$. Thus by Montel's Normality Criterion, $\{f^n\}$ is normal in Ω and hence $\overline{C} - K$ is contained in $F(f)$. □

Now we introduce the notions of fixed points, cycles and stable domains [2, 3, 4, 9, 12, 18].

A solution ζ of the equation $f(z) - z = 0$ is called a *fixed point* of f and $\lambda = f'(\zeta)$ is called its *multiplier* if $\zeta \neq \infty$. If $\zeta = \infty$ we define λ to be the multiplier of the fixed point 0 of the map $z \mapsto \frac{1}{f(\frac{1}{z})}$. If ζ is a fixed point of f^n but not a fixed point of f^m for all m, $0 < m < n$, then ζ is called a *periodic point* of f with period n and $\alpha = \{\zeta, f(\zeta), \cdots, f^{n-1}(\zeta)\}$ is called a *cycle* of f with length n. The multiplier $\lambda(\alpha)$ of a cycle α is defined to be the multiplier of the fixed point ζ of f^n. So by the chain rule $\lambda(\alpha) = (f^n)'(\zeta) = \prod_{j=0}^{n-1} f'(f^j(\zeta))$ at least if $f^j(\zeta) \neq \infty$ for $0 < j < n$. Then the cycle α is called *super-attracting, attracting, indifferent* or *repelling* if $\lambda = 0,\ 0 <\mid \lambda \mid< 1,\ \mid \lambda \mid= 1$ or $\mid \lambda \mid> 1$, respectively. More precisely, the indifferent cycle α is called a *Leau* (or *rationally indifferent*) cycle, *Siegel* cycle or *Cremer* (or *irrationally indifferent*) cycle if $\lambda^m = 1$ for some $m \in \mathbb{N}$, $\alpha \subset F(f)$, or $\alpha \subset J(f)$ and λ is not a root of unit, respectively.

REMARK 2.4. (Super)-attracting cycles lie in the Fatou set, but repelling cycles and rationally indifferent cycles lie in the Julia set.

PROPOSITION 2.5. *Let α be a (super)-attracting cycle of length p. Then each point ζ lies in some stable domain Ω and $f^{np} \to \zeta$ locally uniformly on Ω as $n \to \infty$.*

Proof. Let ζ be a (super)-attracting fixed point of f^p. Then since $|\lambda(\alpha)| < 1$, we can choose δ so that $|\lambda(\alpha)| < \delta < 1$ and a disc $D_{\zeta,\epsilon}$ centered at ζ with radius $\epsilon > 0$ so that

$$\mid f^p(z) - \zeta| = |f^p(z) - f^p(\zeta)| < \delta|z - \zeta| < \delta\epsilon < \epsilon$$

for all $z \in D_{\zeta,\epsilon}$. This shows that

$$|f^{np}(z) - \zeta| < \delta^n|z - \zeta| < \delta^n\epsilon < \epsilon$$

and hence f^{np} maps $D_{\zeta,\epsilon}$ onto itself. So $D_{\zeta,\epsilon} \subset F(f^p)$ and $f^{np} \to \zeta$ locally uniformly on $D_{\zeta,\epsilon}$. But since $F(f) = F(f^p)$, by the Vitali theorem, $f^{np} \to \zeta$ locally uniformly on some stable domain Ω of $F(f)$ which contains ζ. $\square$

A stable domain Ω is called *wandering* if the sets $f^n(\Omega)$ are pairwise disjoint for all $n \geq 0$ and is *periodic* if $f^p(\Omega) = \Omega$ for some $p \in \mathbb{N}$.

It was proved that a rational function of degree at least two has no wandering domain by Sullivan [19] which had been an open problem since the papers of Fatou and Julia [11, 13]. From this, we can say that every stable domain Ω of a rational function f is *eventually periodic,* that is, $f^m(\Omega)$ is periodic for some positive integer m.

The stable domains are classified as follows [4, 5, 9, 19];

FD : A stable fixed domain Ω of f is called a *Fatou domain* if the sequence $\{f^n\}$ converges to a fixed point $\zeta \in \overline{\Omega}$, locally uniformly on Ω. More precisely, a Fatou domain Ω is called

(a) a *Böttcher domain* if ζ is a super-attracting fixed point,

(b) a *Schröder domain* if ζ is an attracting fixed point,

(c) a *Leau domain* if ζ is a rationally indifferent fixed point which lies in $\partial\Omega$.

RD : A stable domain is called a *rotation domain* if none of the limit functions of the sequence $\{f^n|_\Omega\}$ is constant. More precisely, a rotation domain Ω is called

(d) a *Siegel disc* if it is simply connected and contains an indifferent fixed point,

(e) a *Herman ring* if it is doubly connected.

The stable fixed domains are classified by considering those functions which can be expressed as a limit of some subsequence of $\{f^n\}$ on Ω. And a stable fixed domain of f can exactly arise in one of five different ways (a) $-$ (e) ([4]). Furthermore, a rational function of degree d has at most $2(d-1)$ distinct cycles of periodic domains [17].

3. The limit functions on the stable domain

In this section, we consider the properties of which arise as locally uniform limits of subsequences of $\{f^n\}$ in a stable domain Ω.

A function φ is a limit function on a stable domain Ω if there is some subsequence of $\{f^n\}$ which converges locally uniformly to φ on

Ω. The class of limit functions on Ω is denoted by $\mathcal{L}(\Omega)$.

PROPOSITION 3.1. *Suppose that φ and ψ are non-constant limit function in $\mathcal{L}(\Omega)$. Then $\varphi\psi$ is in $\mathcal{L}(\Omega)$ and $\varphi\psi = \psi\varphi$.*

Proof. Let φ and ψ be non-constant limit functions. Then there are sequences $\{n_j\}$ and $\{m_j\}$ of positive integers such that $f^{n_j} \to \varphi$ and $f^{m_j} \to \psi$ locally uniformly. For a given $z \in \Omega$, let K be a compact neighborhood of z on which f^{n_j} and f^{m_j} converge uniformly to φ and ψ, respectively. Since

$$|f^{n_j+m_j} - \varphi\psi| \leq |\psi| \cdot |f^{n_j} - \varphi| + |f^{n_j}| \cdot |f^{m_j} - \psi|,$$

we have $f^{n_j+m_j} \to \varphi\psi$. Similarly, $f^{m_j+n_j} \to \psi\varphi$. Thus $\varphi\psi$ is in $\mathcal{L}(\Omega)$ and $\varphi\psi = \psi\varphi$. □

The zeros of f' and the multiple poles of f are called *critical points.* So f fails to be injective in any neighborhood of a critical point. We denote the set of all critical points of f by $C = C_f$ and the forward orbits of C_f by C_f^+, that is, $C_f^+ = \cup_{n=0}^{\infty} f^n(C_f)$.

Next, we prove the following main theorem.

THEOREM 3.2. *Let f be a rational function of degree at least two. If φ is a constant limit function in some stable domain Ω of $F(f)$, then the value ζ of φ lies in the closure of C_f^+.*

In fact, Theorem 3.2 can be proved without using the Sullivan's theorem. Two rational functions f and g are *conjugate* if and only if there is a rational function $\eta(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ such that $g = \eta f \eta^{-1}$. By the conjugation invariant properties, we may assume that $\zeta = 0$. Suppose that $f^n \to 0$ locally uniformly on some stable domain Ω of $F(f)$ as $n \to \infty$ in some subset N of positive integers. Assume that there exists an $\epsilon > 0$ such that $D_{0,\epsilon} \cap C_f^+ = \phi$, where $D_{0,\epsilon} = \{z \mid \ |z| < \epsilon\}$. Take $z_0 \in \Omega$. For each $n \in N$ with $f^n(z_0) \in D_{0,\epsilon}$, construct a single-valued analytic branch S_n of $(f^n)^{-1}$ in $D_{0,\epsilon}$ such that $S_n \circ f^n(z_0) = z_0$.

Let α, β be disjoint cycles of f with length at least 3. Then S_n cannot map any point $z \in D_{0,\epsilon} \setminus \alpha$ into α. By Montel's Normality Criterion, the family $\{S_n\}$ is normal in $D_{0,\epsilon} \setminus \alpha$. By the same way as above, one can show that $\{S_n\}$ is normal in $D_{0,\epsilon} \setminus \beta$. Thus $\{S_n\}$ is normal in $(D_{0,\epsilon} \setminus \alpha) \cup (D_{0,\epsilon} \setminus \beta) = D_{0,\epsilon}$.

Assume that $S_n \to \Phi$ as $n \to \infty$ on some subset $N \subset \mathbb{N}$. Then

$$z_0 = S_n \circ f^n(z_0) \to \Phi(0)$$

as $n \to \infty$ in some subset $N_1 \subset N$. So $\Phi(0) = z_0$.

For $z_1 \in \Omega$ sufficiently near z_0 and each n with $f^n(z_1) \in D$, we have $f^n(z_1) \to 0$. Since $D_{0,\epsilon}$ has no critical values,

$$S_n \circ f^n(z_1) = z_1 \quad \& \quad S_n \circ f^n(z_1) \to \Phi(0),$$

which contracts the fact that $z_0 \neq z_1$. Thus the proof is completed.

Now we end up with this paper by giving a proof of Theorem 3.2 by using the Sullivan's theorem and the following two lemmas.

Lemma 3.3. ([4]) *If Ω is a stable fixed domain and if there is a constant limit function φ with value ζ, then ζ is a fixed point of f.*

Proof. Let Ω be a stable fixed domain and φ a constant limit function with value ζ. Passing to a subsequence, we may assume that $f^n(z) \to \varphi(z)$ for all $z \in \Omega$. If $z \in \Omega$ then $f(z) \in \Omega$. So $\zeta \in \Omega \cup \partial\Omega$. Thus we have

$$f(z) = f(\lim_{n\to\infty} f^n(z)) = \lim_{n\to\infty} f^n(f(z)) = \varphi(f(z)) = \zeta,$$

as desired. □

LEMMA 3.4. *Let f be holomorphic in some neighborhood U of the fixed point ζ and assume that f^n is defined in U for all n. Then $f^n \to \zeta$ in U if and only if ζ is (super)-attracting.*

Proof. Let $\zeta \in U$ be a (super)-attracting fixed point. Then since $|f'(\zeta)| < 1$, we have

$$|f(z) - \zeta| = |f(z) - f(\zeta)| < \alpha|z - \zeta|$$

for some $0 < \alpha < 1$ and z in some neighborhood $D_{\zeta,\epsilon}$ of ζ. So

$$|f^n(z) - \zeta| \leq \alpha^n|z - \zeta|$$

and so by the Vitalli's theorem, $f^n \to \zeta$ locally uniformly on U. □

Proof of Theorem 3.2. By the Sullivan's theorem, there exists a positive integer m such that $f^m(\Omega) = \Omega$. So Ω is a stable fixed domain of the Fatou set of f^m. Thus Ω must be one of the Böttcher domain, Schröder domain or Leau domain since there is a constant limit function. If φ is a constant limit function with value ζ, then it is also a limit function on Ω of $F(f^l)$ for $1 \leq l \leq m$. So $(f^l)^n \to \zeta$ as $n \to \infty$ in some subsequence. This means that φ is a constant limit function on Ω of $F(f^l)$. Thus by Lemma 3.3, ζ is a fixed point of f^l.

If $\zeta \notin \overline{C_f^+}$, then there exists an open disc $D_{\zeta,\epsilon}$ centered at ζ with radius an $\epsilon > 0$ such that $D_{\zeta,\epsilon} \cap C_f^+ = \phi$. Since $(f^l)^n$ is injective on $D_{\zeta,\epsilon}$ for all n, we have $(f^l)^n(z) \neq \zeta$ for all n and $z \neq \zeta$. Also $(f^l)'$ is never vanished on $D_{\zeta,\epsilon}$. Thus by Lemma 3.4, Ω can be neither a Böttcher domain nor a Schröder domain. Also it cannot be a Leau domain since ζ does not lie in $\partial\Omega$. But it is impossible. Therefore, ζ lies in the closure of C_f^+. □

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