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MULTIPLIER IDEALS ON CR MANIFOLDS

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Abstract. We consider multiplier ideals on CR manifolds, which is associated to Kiremidjian's work on CR embedding problem. Similar to the Kohn's result, we found that the multipliers form a nontrivial radical ideal.

1. Introduction

Let M be a C^∞ manifold of real dimension $2n-1$ and let $T^{\mathbb{C}}M$ be its complexified tangent bundle. A CR-structure on M is given by a complex subbundle $E'' \subset T^{\mathbb{C}}M$ of complex fiber dimension $n-1$ such that $E'' \cap \overline{E''} = \{0\}$ and the Lie bracket of two sections L, L' of E'' over an open subset of M is also a section of E''. We denote the above CR-structure E'' by $T^{0,1}M$ and call its sections as tangent vectors of M of type $(0,1)$. If M is the boundary of a complex n-dimensional manifold M_1, then the complex structure and its deformations induce CR-structure on M.

Kiremidjian considered the small deformation of CR-structure of an embedded CR-manifold M_0 and proved that deformed CR-structure can be extended to a complex structure on a neighborhood N of M_0. His methods are based on estimate which are similar to the one obtained by J. J. Kohn [4].

When Kohn [5] considered subellipticity of $\overline{\partial}$-Neumann problem on pseudoconvex domains, he developed the theory of subelliptic multipliers. He invented an interesting algorithmic procedure for computing

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certain ideals. These ideals, at least in the real analytic case, govern both whether there is a complex analytic variety in the boundary and whether there is a subelliptic estimate.

We consider CR-embedding problem following Kiremidjian's method and develop the similar theory of multipliers.

2. Preliminary

If M is a compact, $2n-1$ dimensional CR-manifold then there is a real $n-1$ plane field, $H \subset TM$ determined by

$$T^{0,1} \oplus T^{1,0} = H \otimes \mathbb{C}.$$

Defining a CR-structure with underlying plane filed H is therefore equivalent to specifying a smooth field of complex structure on the fibers of H. The CR-structure is strictly pseudoconvex if and only if H defines a contact structure. The $\overline{\partial}_b$-operator associated to the CR-structure is defined by

$$\overline{\partial}_b f = df|_{T^{0,1}M};$$

it takes values in the sections of the dual bundle $(T^{0,1})^*$. In order to facilitate the study of the $\overline{\partial}_b$-operator, Kohn and Rossi introduced a "Laplacian" denoted by $\Box_b$. By selecting a one form θ such that $\ker\theta = H$, one can define hermitian metric on $T^{0,1}$ and $(T^{0,1})^*$. The $2n-1$ form $\theta \wedge d\theta^{(n-1)}$ defines a volume form on M and thus we can define L^2-inner products on $C^\infty(M)$ and $C^\infty(M, (T^{0,1}M)^*)$. With these choice we define the L^2-adjoint of $\overline{\partial}_b$, denoted by $\overline{\partial}_b{}^*$, and the associated second order operator $\Box_b = \overline{\partial}_b{}^*\overline{\partial}_b + \overline{\partial}_b\overline{\partial}_b{}^*$

In order to study the deformation of a given CR-structure it is useful to have an explicit parametrization. If $T^{0,1}M'$ is another CR-structure on M, that is close to the given structure $T^{0,1}M$, then its underlying plane field H' is also a contact structure close H. By the theorem of Gray there is a diffeomorphism ϕ of M, close to the identity, such that $\phi_* H' = H$ [2]. The CR-structure $\phi_* T^{0,1}M'$ is another CR-structure on

M close to $T^{0,1}M$ which has H as its underlying contact structure. We consider the perturbations of $T^{0,1}M$ whose underlying contact structure is H. Such perturbations are parametrized by the sections

$$\mathcal{D}(M,\overline{\partial}_b) = \{\Phi \in C^\infty(M, Hom(T^{0,1}M, T^{1,0}M))| \quad ||\Phi||_{L^\infty} < 1\}$$

with

$$^\Phi T^{0.1}_p M = \{\overline{Z} + \Phi_p(\overline{Z}) | \overline{Z} \in T^{0,1}_p M\}$$

for $p \in M$. We denote the $\overline{\partial}_b$-operator associated to CR-structure $^\Phi T^{0.1}M$ by $\overline{\partial}_b{}^\Phi$

Kiremidjian solved the local embedding problem for the deformed CR-structure $^\Phi T^{0.1}M$ by showing the following estimate and using Kohn's work about the existence of Neumann operator.

THEOREM 2.1 (The basic estimate). *Let M be strictly pseudoconvex and $^\Phi T^{0.1}M$ be a CR-structure on M. If $||\Phi||_{C^m}$ is sufficiently small for $m > n+2$, then there exists a constant C, independent of Φ, such that for all $u \in C^\infty(M, (^\Phi T^{0.1}M)^*)$*

$$||u||^2_{\frac{1}{2}} \leq C(||u||^2_0 + ||\overline{\partial}_b{}^\Phi u||^2_0 + ||\overline{\partial}_b{}^{*\Phi} u||^2_0),$$

where $||\ ||_s$ is the Sobolev s-norm over M.

Similar to Kohn's definition of subelliptic multiplier we define multiplier for local CR-embedding problem. Let us denote $D_{0,q}(U)$ for the space of forms of type $(0,q)$ that are in the domain of $\overline{\partial}_b{}^*$ and whose coefficients are in $C^\infty_0(U)$. The formula $||\phi||^2_\epsilon = \sum_{|J|=q} ||\phi_J||^2_\epsilon$ defines the squared Sobolev norm of order ϵ of a form $\phi = \sum_{|J|=q} \phi_J d\overline{z}^J \in D_{0,q}(U)$. As usual the sums are taken over strictly increasing multi-indices.

DEFINITION 2.1. Let M be a CR-manifold and let x be a point in M. Let C^∞_x denote the germs of smooth functions at x. An element $g \in C^\infty_x$ is called a multiplier (on $(0,1)$ forms) if there exist a neighborhood U

and positive constant c, ϵ such that

$$||g\phi||_\epsilon^2 \leq c\left(||\overline{\partial}_b\phi||^2 + ||\overline{\partial}_b{}^*\phi||^2 + ||\phi||^2\right)$$

for all $\phi \in D_{0,1}(U)$. We will denote

$$Q(\phi,\phi) = \left(||\overline{\partial}_b\phi||^2 + ||\overline{\partial}_b{}^*\phi||^2 + ||\phi||^2\right).$$

The collection of multipliers turn out to be a radical ideal. Let us elaborate the notion of radical of an ideal in the ring C_x^∞ [6]. Let J be an ideal in C_x^∞. The radical of J, written $rad_{\mathbb{R}}(J)$, and sometimes called "the real radical" of J, is the collection of germs $g \in C_x^\infty$ such that there is an integer N and an element $f \in J$ for which

$$|g|^N \leq |f|.$$

To preserve some relationship between ideals and varieties, in the real analytic category, one must allow this broader sense of radicals. The Lojasiewicz inequality [6, 7] then becomes a precise analogue of the Nullstellensatz. This inequality can be stated as follows.

THEOREM 2.2. *A (germ of a) real analytic function f vanishes on the zeroes of an ideal J generated by real analytic functions if and only if $f \in rad_{\mathbb{R}}(J)$.*

We consider the local coordinate system $x = (x_1, \ldots, x_{2n-1})$ on an open $U \subset M$. Then one has the Fourier transform $\hat{f}(\xi)$, defined on $C_0^\infty(U)$. One can then define pseudodifferential operator in the usual manner by

$$\Lambda^s f(x) = \frac{1}{(2\pi)^{\frac{2n-1}{2}}} \int_{\mathbb{R}^{2n-1}} e^{ix\xi}\hat{f}(\xi)\left(1+|\xi|^2\right)^{\frac{s}{2}} d\xi.$$

We state formal properties of pseudodifferential operators as follows.

THEOREM 2.3 (Formal properties of pseudodifferential operators).

1. *If P^m is a pseudodifferential operator of order m, then for each s there is a constant c_s such that*

$$||P^m u||_s \le c_s ||u||_{s+m}.$$

2. *If P^m, Q^s denote pseudodifferential operator of the indicated orders, then their commutator is a pseudodifferential operator of order $m+s-1$:*

$$[P^m, Q^s] = T^{m+s-1}.$$

3. *Orders add under composition:*

$$P^m Q^s = T^{m+s}.$$

4. *Let $\overline{P}$ denote the operator whose symbol is obtained by conjugation. Then*

$$(P^m)^* - \overline{(P^m)} = T^{m-1}.$$

3. Main Theorem

We now establish the fundamental properties of multipliers. The collection of all such multipliers form a real radical ideal. The main point is that the estimate in Theorem 2.1 holds on $(0,1)$ forms if and only if the ideal of multipliers is the full ring of germs of smooth functions. For a pseudoconvex domain in $\mathbb{C}^n$, D'Angelo [1] use tangential pseudodifferential operator to prove the theorem about subelliptic multipliers. Replacing tangential objects to the ones on a CR-manifold, we get the exactly same theorem.

Theorem 3.1 (Main Theorem). *The collection of multipliers J_x on $(0,1)$ forms is a radical ideal. In particular,*

$$g \in J_x, \quad |f|^N \le |g| \Rightarrow f \in J_x$$

Proof. First we prove J_x is an ideal. To see this, note that

$$||gu||_\epsilon^2 = \int |\Lambda^\epsilon (gu)|^2$$

$$= \int |g\Lambda^\epsilon u + P^{\epsilon-1}u|^2$$

$$\leq \int |g\Lambda^\epsilon u|^2 + |P^{\epsilon-1}u|^2 + 2|g\Lambda^\epsilon u\overline{P^{\epsilon-1}u}|$$

$$\leq 2\int |g|^2|\Lambda^\epsilon u|^2 + c||u||^2_{\epsilon-1}$$

$$\leq c||u||^2_\epsilon + c||u||^2_{\epsilon-1}$$

$$\leq c||u||^2_\epsilon,$$

where $P^{\epsilon-1}$ is the commutator $[\Lambda^\epsilon, g]$. Replacing g by hf, we get

$$||hfu||_\epsilon \leq c||fu||_\epsilon \leq cQ(u,u).$$

Thus J_x is closed under the multiplication by element of C^∞_x. Similarly,

$$||(f+g)u||^2_\epsilon \leq 2(||fu||^2_\epsilon + ||gu||^2_\epsilon) \leq cQ(u,u).$$

Hence J_x is an ideal.

To show that the ideal of multipliers is a radical ideal, we establish two inequalities. Note that we use the same letter c to denote a positive constant whose value need not be the same for each occurrence.

Lemma 3.2. *If* $|f| \leq |g|$, *then*

$$||f\phi||_\epsilon \leq ||g\phi||_\epsilon + c||\phi||_\epsilon. \tag{1}$$

Proof.

$$||f\phi||^2_\epsilon = (\Lambda^\epsilon f\phi, \Lambda^\epsilon f\phi)$$

$$= ([\Lambda^\epsilon, f]\phi, \Lambda^\epsilon f\phi) + (f\Lambda^\epsilon\phi, \Lambda^\epsilon f\phi)$$

$$= (P^{(\epsilon-1)}\phi, \Lambda^\epsilon f\phi) + (f\Lambda^\epsilon\phi, \Lambda^\epsilon f\phi)$$

$$\leq ||P^{(\epsilon-1)}||||\Lambda^\epsilon f\phi|| + ||f\Lambda^\epsilon\phi||||\Lambda^\epsilon f\phi||.$$

Dividing by $||f\phi||_\epsilon$, and because $P^{\epsilon-1}$ is of order $\epsilon-1$, we obtain the estimate

$$||f\phi||_\epsilon \leq ||\phi||_{\epsilon-1} + ||f\Lambda^\epsilon\phi||.$$

Since $||f\Lambda^\epsilon\phi$ depends only on $|f|$ rather than f, using $|f| \le |g|$, we obtain

$$||f\phi||_\epsilon \le ||\phi||_{\epsilon-1} + ||g\Lambda^\epsilon\phi||.$$

Similarly, we obtain

$$\begin{aligned} ||g\Lambda^\epsilon\phi||^2 &= (g\Lambda^\epsilon\phi, g\Lambda^\epsilon\phi) \\ &= ([g,\Lambda^\epsilon]\phi, g\Lambda^\epsilon\phi) + (\Lambda^\epsilon g\phi, g\Lambda^\epsilon\phi) \\ &= (P^{(\epsilon-1)}\phi, g\Lambda^\epsilon\phi) + (\Lambda^\epsilon g\phi, g\Lambda^\epsilon\phi) \\ &\le ||P^{(\epsilon-1)}||||g\Lambda^\epsilon\phi|| + ||\Lambda^\epsilon g\phi||||g\Lambda^\epsilon\phi|| \end{aligned}$$

and

$$||g\Lambda^\epsilon\phi|| \le ||P^{\epsilon-1}\phi|| + ||\Lambda^\epsilon g\phi||.$$

Combining these two, we get the result. □

Lemma 3.3. *For $m\epsilon \le 1$, we have*

$$||g\phi||_\epsilon^2 \le k||g^m\phi||_{m\epsilon}^2 + c||\phi||^2. \tag{2}$$

Proof. We will use the induction on m. First we suppose that $m = 2$.

$$\begin{aligned} ||g\phi||_\epsilon^2 &= (\Lambda^\epsilon g\phi, \Lambda^\epsilon g\phi) = (\Lambda^{2\epsilon} g\phi, g\phi) \\ &= (g\Lambda^{2\epsilon}\phi, g\phi) + (P^{2\epsilon-1}\phi, g\phi) \\ &= (|g|^2\Lambda^{2\epsilon}\phi, \phi) + (P^{2\epsilon-1}\phi, g\phi) \\ &\le ||\,|g|^2\Lambda^{2\epsilon}\phi||\,||\phi|| + c||\phi||_{2\epsilon-1}||\phi|| \\ &= ||g^2\Lambda^{2\epsilon}\phi||\,||\phi|| + c||\phi||_{2\epsilon-1}||\phi|| \\ &= ||\Lambda^{2\epsilon}g^2\phi + P^{2\epsilon-1}\phi||||\phi|| + c||\phi||_{2\epsilon-1}||\phi|| \\ &\le ||g^2\phi||_{2\epsilon}||\phi|| + c||\phi||_{2\epsilon-1}||\phi|| \\ &\le (sc)||g^2\phi||_{2\epsilon}^2 + (lc)||\phi||^2 + c||\phi||_{2\epsilon-1}||\phi|| \\ &\le (sc)||g^2\phi||_{2\epsilon}^2 + c||\phi||^2, \end{aligned}$$

where (sc) denotes a small constant and (lc) denotes a large constant. In the last line we assumed that $2\epsilon - 1 \leq 0$.

Given the interpolation inequality

$$||g^2\phi||_{2\epsilon}^2 \leq c_1||g^3\phi||_{3\epsilon}||g\phi||_\epsilon + c||\phi||, \tag{3}$$

we have

$$\begin{aligned}
||g\phi||_\epsilon^2 &\leq k||g^2\phi||_{2\epsilon}^2 + c||\phi||^2 \\
&\leq kc_1||g^3\phi||_{3\epsilon}||g\phi||_\epsilon + c||\phi||^2 \\
&\leq \frac{kc_1}{2}||g^3\phi||_{3\epsilon}^2 + \frac{kc_1}{2}||g\phi||_\epsilon^2 + c||\phi||^2
\end{aligned}$$

Assume that $kc_1 < 2$. After subtracting the middle term from each side and multiplying by $2/(2 - kc_1)$, we obtain

$$||g\phi||_\epsilon^2 \leq \frac{kc_1}{2 - kc_1}||g^3\phi||_{3\epsilon}^2 + c||\phi||^2.$$

If we know the value of the constant c_1, then we can choose k arbitrarily small and again make the coefficient in front as small as we wish. It remains to prove the interpolation inequality (3).

$$\begin{aligned}
||g^2\phi||_{2\epsilon}^2 &= (\Lambda^{2\epsilon}g^2\phi, \Lambda^{2\epsilon}g^2\phi) \\
&= (\Lambda^{3\epsilon}g^2\phi, \Lambda^{\epsilon}g^2\phi) \\
&= (\Lambda^{3\epsilon}g^2\phi, g\Lambda^{\epsilon}g\phi) + (\Lambda^{3\epsilon}g^2\phi, P^{\epsilon-1}\phi) \\
&= (\overline{g}\Lambda^{3\epsilon}g^2\phi, \Lambda^{\epsilon}g\phi) + (\Lambda^{2\epsilon}g^2\phi, P^{2\epsilon-1}\phi) \\
&\leq (\overline{g}\Lambda^{3\epsilon}g^2\phi, \Lambda^{\epsilon}g\phi) + (sc)||g^2\phi||_{2\epsilon}^2 + (lc)||\phi||_{2\epsilon-1}^2 \\
&\leq (\overline{g}\Lambda^{2\epsilon}g^2\phi, \Lambda^{2\epsilon}g\phi) + (sc)||g^2\phi||_{2\epsilon}^2 + c||\phi||^2 \\
&= (\Lambda^{\epsilon}\overline{g}\Lambda^{2\epsilon}g^2\phi, \Lambda^{\epsilon}g\phi) + (sc)||g^2\phi||_{2\epsilon}^2 + c||\phi||^2
\end{aligned}$$

We need $2\epsilon - 1 \leq 0$. After subtracting the middle term and multiplying through again, we obtain

$$||g^2\phi||^2_{2\epsilon} \leq \frac{1}{1-(sc)}||\Lambda^\epsilon \overline{g}\Lambda^{2\epsilon}g^2\phi||\,||g\phi||_\epsilon + c||\phi||^2. \tag{4}$$

The first term can be estimated as follows.

$$\begin{aligned} ||\Lambda^\epsilon \overline{g}\Lambda^{2\epsilon}g^2\phi|| &= ||\overline{g}g^2\Lambda^{3\epsilon}\phi + P^{\epsilon-1}|| \\ &\leq ||g^3\Lambda^{3\epsilon}\phi|| + c||\phi||_{3\epsilon-1} \\ &\leq ||g^3\phi||_{3\epsilon} + c||\phi||_{3\epsilon-1}. \end{aligned}$$

Using $3\epsilon - 1 \leq 0$ we estimate the last term. Putting this into 4 and using the result for $m = 2$ with small coefficient, we establish

$$\begin{aligned} ||g^2\phi||^2_{2\epsilon} &\leq \frac{1}{1-(sc)}||g^3\phi||_{3\epsilon}||g\phi|| + \frac{1}{2(1-(sc))}||g\phi||^2_\epsilon + c||\phi||^2 \\ &\leq \frac{1}{1-(sc)}||g^3\phi||_{3\epsilon}||g\phi|| + (sc_2)||g^2\phi||^2_\epsilon + c||\phi||^2. \end{aligned}$$

We subtract again and multiply through. At last we obtain the interpolation inequality with the constant

$$c_1 = \frac{1}{(1-(sc))(1-(sc_2))}$$

which can be made arbitrarily close to the unity from above. This finishes the proof of Lemma 3.3 for $m = 3$. The general case is virtually the same, where the interpolation inequality used is as in Remark 1. □

The fact that the ideal of multipliers is a radical ideal follows immediately from Lemma 3.3 and the definition of real radical. □

Remark 1. The general interpolation inequality is

$$||g^{\frac{a+b}{2}}\phi||^2_{\frac{a+b}{2}\epsilon} \leq c_1||g^a\phi||_{a\epsilon}||g^b\phi||_{b\epsilon} + c||\phi||^2$$

as long as $(a+b)\epsilon \leq 1$.

We also found that the determinant of the Levi form is a multiplier. However, we do not know yet whether we can develop an algorithm similar to Kohn's.

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