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# SKEW COPAIRED BIALGEBRAS

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ABSTRACT. Let $\sigma\colon k \to A\otimes B$ be a skew copairing on $(A,B)$, where $A$ and $B$ are Hopf algebras of the same dimension $n$. Skew dual bases of $A$ and $B$ are introduced. If $\sigma$ is an invertible skew copairing then we can give a 2-cocycle bilinear form $[\sigma]$ on $A\otimes B$ and define a new Hopf algebra.

Let $k$ be a field. All unadorned tensor products are over $k$ and all maps are $k$-linear. For $f,g\in Hom(C,A)$, where $C$ is a coalgebra and $A$ is an algebra, $f*g$ is its convolution product $m_A(f\otimes g)\Delta_C$. Let $\tau\colon V\otimes W\to W\otimes V$ be the twist map given by $\tau(v\otimes w)=w\otimes v$. We use the sigma notation[6]; for $x\in C$, $\Delta(x)=\sum x_{(1)}\otimes x_{(2)}$.

LEMMA 1. *Let $A$ be a bialgebras with invertible linear form under convolution product, $\sigma : k\to A\otimes A$ where $\sigma(1)=\sum\sigma_1(1)\otimes\sigma_2(1)$ and $\sigma^{-1}(1)=\sum\sigma_1^{-1}(1)\otimes\sigma_2^{-1}(1)$. Assume that $\sigma$ satisfies $\sigma_{12}(\Delta\otimes id)\sigma(1)=\sigma_{23}(id\otimes\Delta)\sigma(1)$, where $\sigma_{12}=\sum\sigma_1(1)\otimes\sigma_2(1)\otimes 1$ and $\sigma_{23}=\sum 1\otimes\sigma_1(1)\otimes\sigma_2(1)$, i.e.,*

$$(*)\qquad \sum\sigma_1(1)(\sigma_1(1))_{(1)}\otimes\sigma_2(1)(\sigma_1(1))_{(2)}\otimes\sigma_2(1)$$
$$=\sigma_1(1)\otimes\sigma_1(1)(\sigma_2(1))_{(1)}\otimes\sigma_2(1)(\sigma_2(1))_{(2)}.$$

*Then*

(1) $\sum\sigma_1(1)\sigma_1^{-1}(1)\otimes\sigma_2(1)\sigma_2^{-1}(1)=1\otimes 1,$
$\sum\sigma_1^{-1}(1)\sigma_1(1)\otimes\sigma_2^{-1}(1)\sigma_2(1)=1\otimes 1.$

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(2) $\sum \varepsilon(\sigma_1(1)) \otimes \sigma_2(1) = 1 \otimes 1, \quad \sum \sigma_1(1) \otimes \varepsilon(\sigma_2(1)) = 1 \otimes 1.$

(3) $\sum (\sigma_1^{-1}(1))_{(1)}\sigma_1^{-1}(1) \otimes (\sigma_1^{-1}(1))_{(2)}\sigma_2^{-1}(1) \otimes \sigma_2^{-1}(1)$
$= \sum \sigma_1^{-1}(1) \otimes (\sigma_2^{-1}(1))_{(1)}\sigma_1^{-1}(1) \otimes (\sigma_2^{-1}(1))_{(2)}\sigma_2^{-1}(1).$

(4) $\sum \varepsilon(\sigma_1^{-1}(1)) \otimes \sigma_2^{-1}(1) = 1 \otimes 1, \ \sum \sigma_1^{-1}(1) \otimes \varepsilon(\sigma_2^{-1}(1)) = 1 \otimes 1.$

(5) $\sum \sigma_1(1)\varepsilon(\sigma_1^{-1}(1)) \otimes \sigma_2^{-1}(1)\varepsilon(\sigma_2(1)) = 1 \otimes 1,$
$\sum \sigma_1^{-1}(1)\varepsilon(\sigma_1(1)) \otimes \sigma_2(1)\varepsilon(\sigma_2^{-1}(1)) = 1 \otimes 1.$

(6) *if $A$ has an antipode $S$ then*
$\sum S(\sigma_2(1))\sigma_1(1)S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1) = 1,$
$\sum S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1)S(\sigma_2(1))\sigma_1(1) = 1.$

*Proof.* (1) : $1 \otimes 1 = u_{A\otimes A}\varepsilon_k(1) = (\sigma * \sigma^{-1})(1) = \sigma(1)\sigma^{-1}(1)$ and $1 \otimes 1 = u_{A\otimes A}\varepsilon_k(1) = (\sigma^{-1} * \sigma)(1) = \sigma^{-1}(1)\sigma(1).$

(2) : Apply $id \otimes \varepsilon \otimes id$ to both sides of $(*)$,

$$\sum \sigma_1(1)\sigma_1(1) \otimes \varepsilon(\sigma_2(1)) \otimes \sigma_2(1) = \sum \sigma_1(1) \otimes \varepsilon(\sigma_1(1)) \otimes \sigma_2(1)\sigma_2(1).$$

Multiply $\sum \sigma_1^{-1}(1) \otimes 1 \otimes \sigma_2^{-1}(1)$ to both sides of above,

$$\begin{aligned}\sum \sigma_1(1)\varepsilon(\sigma_2(1)) \otimes 1 \otimes 1 &= \sum \sigma_1(1) \otimes \varepsilon(\sigma_2(1)) \otimes 1 \\ &= \sum 1 \otimes \varepsilon(\sigma_1(1)) \otimes \sigma_2(1) \\ &= \sum 1 \otimes 1 \otimes \varepsilon(\sigma_1(1))\sigma_2(1).\end{aligned}$$

Hence

$$\sum \varepsilon(\sigma_1(1))\sigma_2(1) = 1 = \sum \sigma_1(1)\varepsilon(\sigma_2(1)).$$

(3),(4) : From the definition of $\sigma^{-1}$.

(5) : From (1),(2) and (4).

(6) : $\sum S(\sigma_2(1))\sigma_1(1)S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1)$
$= \sum S(\sigma_2(1))\sigma_1(1)S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1)\varepsilon(\sigma_1^{-1}(1)\sigma_1(1))\varepsilon(\sigma_2^{-1}(1)$
$\sigma_2(1))$
$= \sum S(\sigma_2(1)\varepsilon(\sigma_2^{-1}(1)))\sigma_1(1)\varepsilon(\sigma_1^{-1}(1))S^{-1}(\sigma_2^{-1}(1)\varepsilon(\sigma_2(1)))\sigma_1^{-1}(1)$
$\varepsilon(\sigma_1(1))$
$= 1.$ □

PROPOSITION 1. *Let $A$ be a bialgebra with invertible $\sigma : k \to A\otimes A$. Assume that $\sigma$ satisfies $(*)$. Define $A_\sigma = A$ as an algebra. The coproduct $\Delta_\sigma$ is defined by*

$$\Delta_\sigma(a) = \sum \sigma_1(1)a_{(1)}\sigma_1^{-1}(1) \otimes \sigma_2(1)a_{(2)}\sigma_2^{-1}(1), \quad a \in A.$$

*Then $A_\sigma$ is a bialgebra. Moreover, when $A$ has an antipode $S$, $A_\sigma$ is a Hopf algebra and its antipode is given by*

$$S_\sigma(a) = \sum S(\sigma_2(1))\sigma_1(1)S(a)S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1).$$

*Proof.* For all $a \in A$,

$$\begin{aligned}
&(\Delta_\sigma \otimes id)\Delta_\sigma(a)\\
&= (\Delta_\sigma \otimes id)(\textstyle\sum \sigma_1(1)a_{(1)}\sigma_1^{-1}(1) \otimes \sigma_2(1)a_{(2)}\sigma_2^{-1}(1))\\
&= \textstyle\sum \sigma_1(1)(\sigma_1(1)a_{(1)}\sigma_1^{-1}(1))_{(1)}\sigma_1^{-1}(1) \otimes \sigma_2(1)(\sigma_1(1)a_{(1)}\sigma_1^{-1}(1))_{(2)}\\
&\qquad \sigma_2^{-1}(1) \otimes \sigma_2(1)a_{(2)}\sigma_2^{-1}(1)\\
&= \textstyle\sum \sigma_1(1)(\sigma_1(1))_{(1)}a_{(1)(1)}\,(\sigma_1^{-1}(1))_{(1)}\sigma_1^{-1}(1) \otimes \sigma_2(1)(\sigma_1(1))_{(2)}\\
&\qquad a_{(1)(2)}(\sigma_1^{-1}(1))_{(2)}\sigma_2^{-1}(1) \otimes \sigma_2(1)a_{(2)}\sigma_2^{-1}(1)\\
&= \textstyle\sum \sigma_1(1)a_{(1)}\sigma_1^{-1}(1) \otimes \sigma_1(1)(\sigma_2(1))_{(1)}\,a_{(2)(1)}\,(\sigma_2^{-1}(1))_{(1)}\sigma_1^{-1}(1)\\
&\qquad \otimes \sigma_2(1)(\sigma_2(1))_2\,a_{(2)(1)}\,(\sigma_2^{-1}(1))_{(2)}\sigma_2^{-1}(1)\\
&= \textstyle\sum \sigma_1(1)a_{(1)}\sigma_1^{-1}(1) \otimes \sigma_1(1)(\sigma_2(1)a_{(2)}\sigma_2^{-1}(1))_{(1)}\sigma_1^{-1}(1)\\
&\qquad \otimes \sigma_2(1)(\sigma_2(1)a_{(2)}\sigma_2^{-1}(1))_{(2)}\sigma_2^{-1}(1)\\
&= (id \otimes \Delta_\sigma)(\textstyle\sum \sigma_1(1)a_{(1)}\sigma_1^{-1}(1) \otimes \sigma_2(1)a_{(2)}\sigma_2^{-1}(1))\\
&= (id \otimes \Delta_\sigma)\Delta_\sigma(a),
\end{aligned}$$

where fourth equality follows from $(*)$ and Lemma 1, (3).

For all $a \in A$,

$$\begin{aligned}
&\textstyle\sum \varepsilon(\sigma_1(1)\,a_{(1)}\,\sigma_1^{-1}(1))\sigma_2(1)\,a_{(2)}\sigma_2^{-1}(1)\\
&= \textstyle\sum \varepsilon(\sigma_1(1))\varepsilon(a_{(1)})\varepsilon(\sigma_1^{-1}(1))\sigma_2(1)\,a_{(2)}\sigma_2^{-1}(1)\\
&= \textstyle\sum \varepsilon(\sigma_1(1))\sigma_2(1)\cdot\varepsilon(a_{(1)})a_{(2)}\cdot\varepsilon(\sigma_1^{-1}(1))\sigma_2^{-1}(1)\\
&= a,
\end{aligned}$$

where fourth equality follows from Lemma 1,(2) and (4).

Similarly, $\sum \sigma_1(1)\ a_{(1)}\ \sigma_1^{-1}(1)\varepsilon(\sigma_2(1)\ a_{(2)}\ \sigma_2^{-1}(1)) = a$. Hence $A_\sigma$ is coassociative and counitary with the $\Delta_\sigma$ comultiplication.

For all $a \in A$

$$\begin{aligned}
&\sum S_\sigma(\sigma_1(1)\ a_{(1)}\ \sigma_1^{-1}(1))\sigma_2(1)\ a_{(2)}\ \sigma_2^{-1}(1)\\
&= \sum S_\sigma(\sigma_1^{-1}(1))S_\sigma(\ a_{(1)}\ )S_\sigma(\sigma_1(1))\sigma_2(1)\ a_{(2)}\ \sigma_2^{-1}(1)\\
&= \sum S(\sigma_2(1))\sigma_1(1)S(\sigma_1^{-1}(1))S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1)\cdot S(\sigma_2(1))\sigma_1(1)\\
&\qquad S(a_{(1)})S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1)\cdot S(\sigma_2(1))\sigma_1(1)S(\sigma_1(1))\\
&\qquad S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1)\cdot \sigma_2(1)\ a_{(2)}\ \sigma_2^{-1}(1)\\
&= \sum S(\sigma_2(1))\sigma_1(1)S(\sigma_1^{-1}(1))S(a_1)S(\sigma_1(1))S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1)\\
&\qquad \sigma_2(1)\ a_{(2)}\ \sigma_2^{-1}(1)\\
&= \sum S(\sigma_2(1))\sigma_1(1)S(\sigma_1^{-1}(1))S(a_1)S(\sigma_1(1))S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1)\\
&\qquad \sigma_2(1)\ a_{(2)}\ \sigma_2^{-1}(1)\varepsilon(\sigma_1^{-1}(1)\sigma_1(1))\varepsilon(\sigma_2^{-1}(1))\sigma_2(1))\\
&= \sum S(\sigma_2(1)\varepsilon(\sigma_2^{-1}(1)))\sigma_1(1)\varepsilon(\sigma_1^{-1}(1))S(\sigma_1^{-1}(1))S(a_{(1)})S(\sigma_1(1))\\
&\qquad S^{-1}(\sigma_2^{-1}(1)\varepsilon(\sigma_2(1))\sigma_1^{-1}(1)\varepsilon(\sigma_1(1))\sigma_2(1)\ a_{(2)}\ \sigma_2^{-1}(1)\\
&= \sum S(\sigma_2^{-1}(1))S(a_{(1)})S(\sigma_1(1))\sigma_2(1)\ a_{(2)}\ \sigma_2^{-1}(1)\\
&= \sum S(\sigma_1^{-1}(1))S(a_{(1)})S(\sigma_1(1))\sigma_2(1)\ a_{(2)}\ \sigma_2^{-1}(1)\\
&\qquad \varepsilon(\sigma_1(1)\sigma_1^{-1}(1))\varepsilon(\sigma_2(1)\sigma_2^{-1}(1))\\
&= \sum S(\sigma_1^{-1}(1)\varepsilon(\sigma_1(1)))S(a_{(1)})S(\sigma_1(1)\varepsilon(\sigma_1^{-1}(1)))\sigma_2(1)\varepsilon(\sigma_2^{-1}(1))\\
&\qquad a_{(2)}\sigma_2^{-1}(1)\varepsilon(\sigma_2(1))\\
&= \sum S(a_{(1)})\ a_{(2)}
\end{aligned}$$

$= \varepsilon(a)\ 1_A$, where third, sixth and ninth equalities follow from Lemma 1,(5) and fourth and seventh equalities follow from Lemma 1,(1).

Similarly,

$$\sum \sigma_1(1)\ a_{(1)}\ \sigma_1^{-1}(1)S_\sigma(\sigma_2(1))\ a_{(2)}\ \sigma_2^{-1}(1)) = \varepsilon(a)\ 1_A, \qquad \forall a \in A.$$

Hence $S_\sigma$ is antipode of $A_\sigma$. For all $a,\ b \in A$,

$$\begin{aligned}
\Delta_\sigma(ab) &= \sigma_1(1)(ab)_{(1)}\sigma_1^{-1}(1)\ \otimes\ \sigma_2(1)(ab)_{(2)}\sigma_2^{-1}(1)\\
&= \sigma_1(1)a_{(1)}b_{(1)}\sigma_1^{-1}(1)\ \otimes\ \sigma_2(1)a_{(2)}b_{(2)}\sigma_2^{-1}(1).
\end{aligned}$$

$\Delta_\sigma(a)\Delta_\sigma(b) = (\sigma_1(1)\ a_{(1)}\ \sigma_1^{-1}(1) \otimes \sigma_2(1)\ a_{(2)}\ \sigma_2^{-1}(1))(\sigma_1(1)\ b_{(1)}$

$$\sigma_1^{-1}(1) \otimes \sigma_2(1)\, b_{(2)}\, \sigma_2^{-1}(1))$$
$$= \sigma_1(1)\, a_{(1)}\, \sigma_1^{-1}(1)\sigma_1(1)\, b_{(1)}\, \sigma_1^{-1}(1) \otimes \sigma_2(1)$$
$$a_{(2)}\, \sigma_2^{-1}(1)\sigma_2(1)\, b_{(2)}\, \sigma_2^{-1}(1)$$
$$= \sigma_1(1)a_{(1)}b_{(1)}\sigma_1^{-1}(1) \otimes \sigma_2(1)a_{(2)}b_{(2)}\sigma_2^{-1}(1).$$

Hence $\Delta_\sigma(ab) = \Delta_\sigma(a)\Delta_\sigma(b), \quad \forall a,\ b \in A.$ Trivially $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$. □

Let $A$ be a bialgebra, recall the definition of quasitriangular bialgebra in [3] and [4], which is the dual concept of a co-quasitriangular bialgebra in [1].

Definition 1. *A quasitriangular bialgebra* over $k$ is a pair $(A,\ \sigma)$ where $A$ is a bialgebra over $k$ and $\sigma : k \longrightarrow A \otimes A,\ 1 \mapsto \sum \sigma_1(1) \otimes \sigma_2(1)$ is an invertible (with respect to convolution product) linear form such that the following conditions hold :

(*i*) $\sigma(1)\Delta(a) = \Delta^{op}(a)\sigma(1)$ for all $a \in A$, i.e.,

$$\sum \sigma_1(1)a_1 \otimes \sigma_2(1)a_2 = \sum a_2\sigma_1(1) \otimes a_1\sigma_2(1) \cdots\cdots\cdots\cdots (1)$$

(*ii*) $(\Delta \otimes id)\sigma(1) = \sigma_{13}\sigma_{23}$, where $\sigma_{13} = \sum \sigma_1(1) \otimes 1 \otimes \sigma_2(1)$ and $\sigma_{23} = \sum 1 \otimes \sigma_1(1) \otimes \sigma_2(1)$, i.e. ,

$$\sum(\sigma_1(1))_{(1)} \otimes (\sigma_1(1))_{(2)} \otimes \sigma_2(1) = \sum \sigma_1(1) \otimes \sigma_1(1) \otimes \sigma_2(1)$$
$$\sigma_2(1) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

(*iii*) $(id \otimes \Delta)\sigma(1) = \sigma_{13}\sigma_{12}$, where $\sigma_{12} = \sum \sigma_1(1) \otimes \sigma_2(1) \otimes 1$, i.e.,

$$\sum \sigma_1(1) \otimes (\sigma_2(1))_{(1)} \otimes (\sigma_2(1))_{(2)} = \sum \sigma_1(1)\sigma_1(1) \otimes \sigma_2(1) \otimes$$
$$\sigma_2(1) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (3)$$

A is triangular if also $\sigma^{-1}(1) = \tau(\sigma(1))$.

Proposition 2 [2]. *If $(A,\ \sigma)$ is quasitriangular, then*

$$(*) \qquad \sum \sigma_1(1)(\sigma_1(1))_{(1)} \otimes \sigma_2(1)(\sigma_1(1))_{(2)} \otimes \sigma_2(1)$$
$$= \sum \sigma_1(1) \otimes \sigma_1(1)(\sigma_2(1))_{(1)} \otimes \sigma_2(1)(\sigma_2(1))_{(2)}$$

*i.e.*,

$\sigma_{12}\sigma_{13}\sigma_{23} = \sigma_{23}\sigma_{13}\sigma_{12}$ , $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (4)

THEOREM 1. *If $A$ is a cocommutative quasitriangular with non-trivial invertible $\sigma : k \longrightarrow A \otimes A$ then $(A_\sigma,\ \hat{\sigma}(1))$ is a triangular bialgebra, where $\hat{\sigma}(1)$ is defined by*

$$\hat{\sigma}(1) = \sum \sigma_2(1)\sigma_1^{-1}(1) \otimes \sigma_1(1)\sigma_2^{-1}(1).$$

*Proof.*

$$\begin{aligned}\tau(\hat{\sigma}(1)) \cdot \hat{\sigma}(1) &= \sum \sigma_1(1)\sigma_2^{-1}(1)\sigma_2^{-1}(1)\sigma_2(1)\sigma_1^{-1}(1) \\ &\qquad \otimes \sigma_2(1)\sigma_1^{-1}(1)\sigma_1(1)\sigma_2^{-1}(1) \\ &= \sum \sigma_1(1)\sigma_1^{-1}(1) \otimes \sigma_2(1)\sigma_2^{-1}(1) \\ &= 1 \otimes 1\end{aligned}$$

by Lemma 1,(1). Similarly,

$\hat{\sigma}(1)\tau(\hat{\sigma}(1)) = 1 \otimes 1$. Therefore

$$(\hat{\sigma}(1))^{-1} = \tau(\hat{\sigma}(1)).$$

$$\begin{aligned}\hat{\sigma}(1)\Delta_\sigma(a) &= (\textstyle\sum \sigma_2(1)\sigma_1^{-1}(1) \otimes \sigma_1(1)\sigma_2^{-1}(1))(\textstyle\sum \sigma_1(1)a_1\sigma_1^{-1}(1) \otimes \\ &\quad \sigma_2(1)a_2\sigma_2^{-1}(1)) \\ &= \textstyle\sum \sigma_2(1)\sigma_1^{-1}(1)\sigma_1(1)a_1\sigma_1^{-1}(1) \otimes \sigma_1(1)\sigma_2^{-1}(1)\sigma_2(1)a_2 \\ &\quad \sigma_2^{-1}(1) \\ &= \textstyle\sum \sigma_2(1)a_1\sigma_1^{-1}(1) \otimes \sigma_1(1)a_2\sigma_2^{-1}(1).\end{aligned}$$

$$\begin{aligned}\Delta_\sigma^{op}(a)\hat{\sigma}(1) &= (\textstyle\sum \sigma_2(1)a_2\sigma_2^{-1}(1) \otimes \sigma_1(1)a_1\sigma_1^{-1}(1))(\textstyle\sum \sigma_2(1)\sigma_1^{-1}(1) \\ &\quad \otimes\sigma_1(1)\sigma_2^{-1}(1)) \\ &= \textstyle\sum \sigma_2(1)a_2\sigma_2^{-1}(1)\sigma_2(1)\sigma_1^{-1}(1) \otimes \sigma_1(1)a_1\sigma_1^{-1}(1)\sigma_1(1) \\ &\quad \sigma_2^{-1}(1) \\ &= \textstyle\sum \sigma_2(1)a_2\sigma_1^{-1}(1) \otimes \sigma_1(1)a_1\sigma_2^{-1}(1)\end{aligned}$$

$$= \sum \sigma_2(1)a_1\sigma_1^{-1}(1) \otimes \sigma_1(1)a_2\sigma_2^{-1}(1),$$

where the last equality holds since $A$ is cocommutative. Therefore

$$\Delta_\sigma^{op}(a)\hat{\sigma}(1) = \hat{\sigma}(1)\Delta_\sigma(a).$$

$$\begin{aligned}
(\Delta \otimes id)\hat{\sigma}(1) &= (\Delta_\sigma \otimes id)(\textstyle\sum \sigma_2(1)\sigma_1^{-1}(1) \otimes \sigma_1(1)\sigma_2^{-1}(1)) \\
&= \textstyle\sum \sigma_1(1)(\sigma_2(1)\sigma_1^{-1}(1))_{(1)}\sigma_1^{-1}(1) \otimes \sigma_2(1) \\
&\quad (\sigma_2(1)\sigma_1^{-1}(1))_{(2)}\sigma_2^{-1}(1) \otimes \sigma_1(1)\sigma_2^{-1}(1) \\
&= \textstyle\sum \sigma_1(1)(\sigma_2(1))_{(1)}(\sigma_1^{-1}(1))_{(1)}\sigma_1^{-1}(1) \otimes \sigma_2(1) \\
&\quad (\sigma_2(1))_{(2)}(\sigma_1^{-1}(1))_{(2)}\sigma_2^{-1}(1) \otimes \sigma_1(1)\sigma_2^{-1}(1) \\
&= \textstyle\sum \sigma_1(1)(\sigma_2(1))_{(1)}\sigma_1^{-1}(1) \otimes \sigma_2(1)(\sigma_2(1))_{(2)} \\
&\quad (\sigma_2^{-1}(1))_{(1)}\sigma_1^{(-1)}(1) \otimes \sigma_1(1)(\sigma_2^{-1}(1))_{(2)}\sigma_2^{-1}(1) \\
&= \textstyle\sum \sigma_1(1)\sigma_2(1)\sigma_1^{-1}(1) \otimes \sigma_2(1)\sigma_2(1)(\sigma_2^{-1}(1))_{(1)} \\
&\quad \sigma_1^{-1}(1) \otimes \sigma_1(1)\sigma_1(1)(\sigma_2^{-1}(1))_{(2)}\sigma_2^{-1}(1) \\
&= \textstyle\sum \sigma_1(1)\sigma_2(1)\sigma_1^{-1}(1) \otimes \sigma_2(1)(\sigma_2^{-1}(1))_{(2)}\sigma_2(1) \\
&\quad \sigma_1^{-1}(1) \otimes \sigma_1(1)(\sigma_2^{-1}(1))_{(1)}\sigma_1(1)\sigma_2^{-1}(1) \\
&= \textstyle\sum \sigma_1(1)\sigma_2(1)\sigma_1^{-1}(1) \otimes (\sigma_2^{-1}(1))_{(1)}\sigma_2(1)\sigma_2(1) \\
&\quad \sigma_1^{-1}(1) \otimes (\sigma_2^{-1}(1))_{(2)}\sigma_1(1)\sigma_1(1)\sigma_2^{-1}(1) \\
&= \textstyle\sum \sigma_2(1)\sigma_1(1)\sigma_1^{-1}(1) \otimes (\sigma_2^{-1}(1))_{(1)}\sigma_2(1)\sigma_2(1) \\
&\quad \sigma_1^{-1}(1) \otimes (\sigma_2^{-1}(1))_{(2)}\sigma_1(1)\sigma_1(1)\sigma_2^{-1}(1) \\
&= \textstyle\sum \sigma_2(1)\sigma_1(1)\sigma_1^{-1}(1) \otimes \sigma_2(1)(\sigma_2^{-1}(1))_{(2)}\sigma_2(1) \\
&\quad \sigma_1^{-1}(1) \otimes \sigma_1(1)(\sigma_2^{-1}(1))_{(1)}\sigma_1(1)\sigma_2^{-1}(1) \\
&= \textstyle\sum \sigma_2(1)\sigma_1(1)\sigma_1^{-1}(1)\sigma_1^{-1}(1) \otimes \sigma_2(1)\sigma_2^{-1}(1)\sigma_2(1) \\
&\quad \sigma_1^{-1}(1) \otimes \sigma_1(1)\sigma_2^{-1}(1)\sigma_1(1)\sigma_2^{-1}(1) \\
&= \textstyle\sum \sigma_2(1)\sigma_1^{-1}(1) \otimes \sigma_2(1)\sigma_1^{-1}(1) \otimes \sigma_1(1)\sigma_2^{-1}(1)\sigma_1(1) \\
&\quad \sigma_2^{-1}(1) \\
&= \hat{\sigma}_{13}\hat{\sigma}_{23},
\end{aligned}$$

where fourth equality follows from Proposition 2 and Lemma 1, (3) and fifth equality follows by multiplying $1 \otimes \sigma_1(1) \otimes \sigma_2(1)$ to both sides of (3) and sixth, seventh and ninth equalities follow from (1) and eighth equality follows from (4) and tenth equality follows from (3).

Similarly,

$$(id \otimes \Delta_\sigma)\hat{\sigma}(1) = \hat{\sigma}_{13}\hat{\sigma}_{23},$$

as desired. □

Let $H$ and $K$ be bialgebras, recall the definition of skew copairing in [5], which is the dual concept of skew pairing in [7].

Definition 2. Let $H$ and $K$ be bialgebras. We say that $H$ and $K$ are *copaired* if there exists a $k$-linear map $\sigma : k \to H \otimes K$, $\sigma(1) = \sum \sigma_1(1) \otimes \sigma_2(1)$ *(called the skew copairing)* such that the diagrams below commute :

$$\begin{array}{ccccc}
k & \xrightarrow{\sigma} & H \otimes K & \xrightarrow{id\otimes\Delta_k} & H \otimes K \otimes K \\
\Delta_k \downarrow & & & & \uparrow \mu_H\otimes id\otimes id \\
k \otimes k & \xrightarrow{\sigma\otimes\sigma} & H \otimes K \otimes H \otimes K & \xrightarrow{id\otimes\tau\otimes id} & H \otimes H \otimes K \otimes K
\end{array}$$

$$\begin{array}{ccccc}
k & \xrightarrow{\sigma} & H \otimes K & \xrightarrow{\Delta_H\otimes id} & H \otimes H \otimes K \\
\Delta_k \downarrow & & & & \uparrow id\otimes id\otimes\mu_K \\
k \otimes k & \xrightarrow{\sigma\otimes\sigma} & H \otimes K \otimes H \otimes K & \xrightarrow{id\otimes\tau\otimes id} & H \otimes H \otimes K \otimes K
\end{array}$$

$$\begin{array}{ccccc}
k & \xrightarrow{id} & k & \xrightarrow{id} & k \\
\downarrow 1_H\otimes & & \downarrow \sigma & & \downarrow \otimes 1_K \\
H \otimes k & \xleftarrow[id\otimes\epsilon_K]{} & H \otimes K & \xrightarrow{\epsilon_H\otimes id} & k \otimes K
\end{array}$$

The commutativity of the diagrams above can be expressed equationally in the following way:

$$\sum \sigma_1(1)\otimes(\sigma_2(1))_{(1)}\otimes(\sigma_2(1))_{(2)} = \sum \sigma_1(1)\sigma_1(1)\otimes\sigma_2(1)\otimes\sigma_2(1)\cdot\cdot(1)'$$

$\sum(\sigma_1(1))_{(1)}\otimes(\sigma_1(1))_{(2)}\otimes\sigma_2(1) = \sum\sigma_1(1)\otimes\sigma_1(1)\otimes\sigma_2(1)\sigma_2(1)\cdot\cdot(2)^{'}$
$\sum\varepsilon(\sigma_1(1))\otimes\sigma_2(1) = 1\otimes 1,\ \ \sum\sigma_1(1)\otimes\varepsilon(\sigma_2(1)) = 1\otimes 1\cdots\cdots\cdots(3)^{'}$
where $\sigma(1) = \sum\sigma_1(1)\otimes\sigma_2(1)$.

Note that condition $(3)^{'}$ follows from $(1)^{'}$ and $(2)^{'}$ if $\sigma$ is invertible in $Hom(k, H\otimes K)$.

Example 1. Let $V$ be a finite dimensional vector space with basis $\{v_i\}$. The dual vector space $V^*$ has the dual basis $\{v^i\}$. Let us express the isomorphism

$$\lambda_{U,V} : V\otimes U^* \to Hom(U,V), \quad v\otimes\alpha \mapsto \lambda_{U,V}(v\otimes\alpha)$$

where $\lambda_{U,V}(v\otimes\alpha)(u) = \alpha(u)v, \quad \forall u\in U$. Let $f : U\to V$ be a linear map. Using bases for $U$ and $V$, and have

$$f(u_j) = \sum_i f^i_j v_i$$

for some family $(f^i_j)_{ij}$ of scalars. It is easily check that

$$f \ = \ \lambda_{U,V}(\sum_{ij} f^i_j v_i \ \otimes \ u^j).$$

In particular, taking for $f$ the identity of $V$, we get

$$id_V \ = \ \lambda_{V,V}(\sum_i v_i \ \otimes \ v^i).$$

This allows us to define the *coevaluation map* of any finite dimensional vector space $V$ as the linear map $\delta_V \ : \ k\to V\otimes V^*$ defined by

$$\delta_V(1) \ = \ \lambda^{-1}_{V,V}(id_V) = \sum_i v_i\otimes v^i.$$

Let $H$ be a finite dimensional Hope algebra. Define $\sigma$ as the coevaluation map

$$\sigma \ : \ k\to H^{op}\otimes H^*, \quad \sigma(1) \ = \ \sum h_i \ \otimes \ h^*_i.$$

Then we see that $\sigma$ is skew copairing on $H^{op}$ and $H^*$.

EXAMPLE 2. Let $H = H_4$ be Sweedler's four-dimensional Hopf algebra over $k$ with char $k \neq 2$. As an algebra over $k$, $H$ is generated by $g$ and $x$ with relations

$$g^2 = 1, \ \ x^2 = 0, \ \ xg = -gx.$$

The coalgebra structure and antipode are determine by

$$\Delta(g) \ = \ g \otimes g, \qquad \Delta(x) \ = \ (x \otimes g) + (1 \otimes x),$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(g) = g = g^{-1}, \quad S(x) = gx.$$

$H$ has a basis $\{1, g, x, gx\}$. Let $A = kZ_2$, where $Z_2$ is written multiplicatively as $\{1, a\}$. Now let

$$\sigma(1) = \frac{1}{2}(1 \otimes 1 + 1 \otimes a + g \otimes 1 - g \otimes a) \in H \otimes A.$$

Then one can easily check that $\sigma$ is a skew copairing of $(H, \ A)$ with $\sigma^{-1} = \sigma$.

PROPOSITION 3. *Let $\sigma$ be a skew copairing on $(A, \ B)$, where $A$ and $B$ are bialgebras.*

*(i). If $\sigma$ is invertible in $Hom_k(k, \ A \otimes B)$ which has the convolution product, then we have*

(1) $\sigma(1)\, \sigma^{-1}(1) \ = \ 1 \ \otimes \ 1, \qquad \sigma^{-1}(1)\sigma(1) \ = \ 1 \ \otimes \ 1,$

(2) $\sum(\sigma_1^{-1}(1))_{(1)} \otimes (\sigma_1^{-1}(1))_{(2)} \otimes \sigma_2^{-1}(1)$
$\qquad = \sum \sigma_1^{-1}(1) \otimes \sigma_1^{-1}(1) \otimes \sigma_2^{-1}(1)\sigma_2^{-1}(1),$

(3) $\sum \sigma_1^{-1}(1) \otimes (\sigma_2^{-1}(1))_{(1)} \otimes (\sigma_2^{-1}(1))_{(2)}$
$\qquad = \sum \sigma_1^{-1}(1)\sigma_1^{-1}(1) \otimes \sigma_2^{-1}(1) \otimes \sigma_2^{-1}(1),$

(4) $\sum \varepsilon(\sigma_1^{-1}(1)) \otimes \sigma_2^{-1}(1) \ = \ 1 \ \otimes \ 1,$
$\quad \sum \sigma_1^{-1}(1) \otimes \varepsilon(\sigma_2^{-1}(1)) \ = \ 1 \ \otimes \ 1$

*where* $\sigma^{-1}(1) = \sum \sigma_1^{-1}(1) \otimes \sigma_2^{-1}(1)$.

*(ii). If both* $A$ *and* $B$ *are Hope algebras, we have*

(1) $\sigma^{-1}(1) = \sum S_A(\sigma_1(1)) \otimes \sigma_2(1)$,

(2) $\sigma(1) = \sum \sigma_1^{-1}(1) \otimes S_B(\sigma_2^{-1}(1))$

*where* $S_A$, $S_B$ *are antipodes of* $A$ *and* $B$ *respectively.*

The proof is easy.

Definition 3. Let $A$ and $B$ finite dimensional Hopf algebras and $dim_k A = dim_k B = n$ and $\sigma$ be a skew copairing on $(A,\ B)$. If $\{a_i\}_{i=1,2,\cdots,n}$ is a basis of $A$ and $\{b_i\}_{i=1,2,\cdots,n}$ is a basis of $B$, and $\{a_i^*\}_{i=1,2,\cdots,n}$ and $\{b_i^*\}_{i=1,2,\cdots,n}$ are dual bases of $\{a_i\}$ and $\{b_i\}$ respectively, $\{a_i\}$ and $\{b_i\}$ are called *skew dual bases* of $A$ and $B$ if

$$\sum a_i^*(\sigma_1(1))\ \ b_j^*(\sigma_2(1)) = \delta_{ij}, \quad i,\ j = 1, 2, \cdots, n.$$

Example 3. Let $H$ be a finite dimensional Hope algebra. Define $\sigma$ as the coevaluation map $\sigma\ :\ k \rightarrow H^{op} \otimes H^*$, $\sigma(1) = \sum h_i \otimes h_i^*$. Then we see that $\sigma$ is an invertible skew copairing on $H^{op}$ and $H^*$. Since $H^{op}$ is a Hopf algebra with antipode $S^{-1}$, by Proposition 2, $\sigma^{-1}(1) = \sum S^{-1}(h_i) \otimes h_i^*$. By the definition $\sum h_i^*(h_k)h_j(h_k^*) = \delta_{ij}$. So the dual basis of $H^{op}$ and $H^*$ is the skew dual basis of them.

In the remainder of this paper, we assume that $B$ and $H$ are finite dimensional and $dim_k A = dim_k B = n$.

Proposition 4. *Let* $(B,\ H,\ \sigma)$ *be a skew copair of Hopf algebras,* $\{b_i\}$ *and* $\{h_i\}$ *be the skew dual basis of* $B$ *and* $H$. *Then*

*(i)* $\sum b^*(\sigma_1(1))h_i^*(\sigma_2(1))b_i^* = b^*, \qquad \forall b \in B,$

*(ii)* $\sum b_i^*(\sigma_1(1))h^*(\sigma_2(1))h_i^*, \qquad \forall h \in H.$

*Proof.* $(i)$: $b^* = \sum b^*(b_j)b_j^*$ *where* $b_j^*(b_i) = \delta_{ij}$.

$$\begin{aligned}\sum b^*(\sigma_1(1))h_i^*(\sigma_2(1))b_i^* &= \sum b^*(b_j)b_j^*(\sigma_1(1))h_i^*(\sigma_2(1))b_i^* \\ &= \sum b^*(b_i)b_i^*\end{aligned}$$

$$= b^*, \quad b \in B.$$

*(ii)*: Similarly, $\sum b_i^*(\sigma_1(1))h^*(\sigma_2(1))h_i^* = h^*, \quad \forall\ h \in H$, as desired. □

DEFINITION 4. Let $(B,\ H,\ \sigma)$ be a skew copair of Hopf algebra, $\{b_i\}$ and $\{h_i\}$ the bases of $B$ and $H$ respectively. We call the $n \times n$ matrix $A = (b_i^*(\sigma_1(1))h_j^*(\sigma_2(1))$ the *measure matrix of the skew copairing* $\sigma$ for the pair of bases $\{b_i\}$ and $\{h_i\}$.

DEFINITION 5. Let $(B,\ H,\ \sigma)$ be a skew copair of Hopf algebras. The skew copairing $\sigma$ is called *non-degenerate,* if the measure matrix of skew copairing $\sigma$ for any pair of bases $\{b_i\}$ and $\{h_i\}$ is invertible.

Following Definitions 4 and 5, we have

PROPOSITION 5. *Let $(B,\ H,\ \sigma)$ be a skew copair of Hopf algebra, $\{b_i\}$ and $\{h_i\}$ be the skew dual bases for $(B,\ H,\ \sigma)$, then the measure matrix of the skew copairing $\sigma$ for the pair of bases $\{b_i\}$ and $\{h_i\}$ is $I_n$.*

THEOREM 2. *Let $(B,\ H,\ \sigma)$ be a skew copair of Hopf algebras, $\sigma$ be non-degenerate, then there exists bases $\{b_i\}$ and $\{h_i\}$ which are skew dual bases of $B$ and $H$.*

*Proof.* Let $\{b_i^{'}\}$ and $\{h_i^{'}\}$ be base of $B$ and $H$. Since $\sigma$ is non-degenerate, then by definition, the measure matrix $A$ of $\sigma$ for the pair of bases $\{b_i^{'}\}$ and $\{h_i^{'}\}$ is invertible. By the theory of linear algebra, there exists $n \times n$ matrices $P$ and $Q$ such that

$$P\,A\,Q = I_n.$$

Let

$$P\begin{pmatrix} b_1^{'} \\ b_2^{'} \\ \cdot \\ \cdot \\ \cdot \\ b_n^{'} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix}, \qquad Q^T\begin{pmatrix} h_1^{'} \\ h_2^{'} \\ \cdot \\ \cdot \\ \cdot \\ h_n^{'} \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \cdot \\ \cdot \\ \cdot \\ h_n \end{pmatrix}$$

where $Q^T$ is the transposed matrix of $P$. Then $\{b_i\}_{i=1,2,\cdots,n}$ and $\{h_i\}_{i=1,2,\cdots,n}$ are bases of $B$ and $H$, and are skew dual basis for $(B,\ H,\ \sigma)$. □

EXAMPLE 4. Define for $\alpha \in k$, $\sigma_\alpha : k \to H_4 \otimes H_4$ by

$$\sigma_\alpha(1) = (1\otimes 1 + 1\otimes g + g\otimes 1 - g\otimes g) + \alpha(x\otimes x - x\otimes gx + gx\otimes x + gx\otimes gx).$$

Then it is easy to see that $(H_4,\ H_4,\ \sigma_\alpha)$ is a skew copaired Hopf algebras. For the pair of bases $\{1, g, x, gx\}$. and $\{1, g, x, gx\}$, the measure matrix of $\sigma_\alpha$ is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & \alpha & -\alpha \\ 0 & 0 & \alpha & \alpha \end{pmatrix}.$$

$A$ is invertible whenever $\alpha \neq 0$, i.e , $\sigma_\alpha$is non degenerate. By the theory of elementary transformation in linear algebra, we can get the invertible matrices $P$, $Q$ such that $P\,A\,Q = I_n$ where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & -\frac{1}{2\alpha} & \frac{1}{2\alpha} \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Taking

$$Q^T \begin{pmatrix} 1 \\ g \\ x \\ gx \end{pmatrix} = \begin{pmatrix} 1 \\ g-1 \\ x \\ x+gx \end{pmatrix}, \quad P \begin{pmatrix} 1 \\ g \\ x \\ gx \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} - \frac{g}{2} \\ \frac{x}{\alpha} \\ -\frac{x}{2\alpha} + \frac{gx}{2\alpha} \end{pmatrix}.$$

Then by Theorem 2, $\{1, g-1, x, x+gx\}$, and $\{1, \frac{1}{2}-\frac{g}{2}, \frac{x}{\alpha}, -\frac{x}{2\alpha}+\frac{gx}{2\alpha}\}$, are skew dual bases of $H_4$.

LEMMA 2. *Let $\sigma$ be an invertible skew copairing on $(B,\ H)$ and let $A = B \otimes H$. Then the bicolinear from $[\sigma]$ on $A$ defined by*

$$[\sigma] : k \longrightarrow (B \otimes H) \otimes (B \otimes H),$$

$1 \mapsto \sum [\sigma]_1(1) \otimes [\sigma]_2(1) = \sum (\sigma_1(1) \otimes 1) \otimes (1 \otimes \sigma_2(1))$ *satisfies $(*)$ with inverse* $[\sigma]^{-1}(1) = \sum (\sigma_1^{-1}(1) \otimes 1) \otimes (1 \otimes \sigma_2^{-1}(1))$.

*Proof.* $\sum [\sigma]_1(1)([\sigma]_1)_{(1)} \otimes [\sigma]_2(1)([\sigma]_1(1))_{(2)} \otimes [\sigma]_2(1)$
$= \sum (\sigma_1(1) \otimes 1)(\sigma_1(1) \otimes 1)_{(1)} \otimes (1 \otimes \sigma_2(1))(\sigma_1(1) \otimes 1)_{(2)} \otimes (1 \otimes \sigma_2(1))$
$= \sum (\sigma_1(1)(\sigma_1(1))_{(1)} \otimes 1) \otimes (\sigma_1(1)_{(2)} \otimes \sigma_2(1)) \otimes (1 \otimes \sigma_2(1))$
$= \sum (\sigma_1(1)\sigma_1(1) \otimes 1) \otimes (\sigma_1(1) \otimes \sigma_2(1)) \otimes (1 \otimes \sigma_2(1))\sigma_2(1))$
$= \sum (\sigma_1(1) \otimes 1) \otimes (\sigma_1(1) \otimes (\sigma_2(1))_{(1)}) \otimes (1 \otimes \sigma_2(1)(\sigma_2(1))_{(2)})$

$$= \sum(\sigma_1(1)\otimes 1)\otimes(\sigma_1(1)\otimes 1)(1\otimes\sigma_2(1))_{(1)})\otimes(1\otimes\sigma_2(1))(1\otimes\sigma_2(1))_{(2)}$$
$$= \sum[\sigma]_1(1)\otimes[\sigma]_1(1)([\sigma]_2(1))_{(1)}\otimes[\sigma]_2(1)([\sigma]_2(1))_{(2)},$$

where third equality follows from $(2)'$ and fourth equality follows from $(2)'$. □

We say 2 - cocycle $[\sigma]$ *is associated with* $\sigma$.

THEOREM 3. *Let $\sigma$ be an invertible skew copairing of bialgebras and let $[\sigma]$ be the 2 - cocycle in Lemma 2. Then*

*(i) $A_{[\sigma]} = B\otimes H$ as algebra.*

*(ii) The coproduct $\Delta_{[\sigma]}$ of $A_{[\sigma]}$ is given by*

$$\Delta_{[\sigma]}(b\otimes h) = \sum(\sigma_1(1)b_{(1)}\sigma_1^{-1}(1)\otimes h_{(1)})\otimes(b_{(2)}\otimes\sigma_2(1)h_{(2)}\sigma_2^{-1}(1))$$

*for all $b\in B$, $h\in H$.*

*(iii) If both $B$ and $H$ are Hopf algebras, then $A_{[\sigma]}$ is a Hopf algebras with antipode*

$$S_{[\sigma]}(b\otimes h) = \sum\sigma_1(1)S_B(b)\sigma_1^{-1}(1)\otimes S_H(\sigma_2(1))S_H(h) S_H^{-1}(\sigma_2^{-1}(1)).$$

*Proof.* $(i)$: It follows by the definition of $A_\sigma$.

$(ii)$ : *Let* $[\sigma] : k\to(B\otimes H)\otimes(B\otimes H),\ 1\mapsto\sum[\sigma]_1(1)\otimes[\sigma]_2(1)$.

$$\begin{aligned}\Delta_{[\sigma]}(b\otimes h) &= \sum[\sigma]_1(1)(b\otimes h)_{(1)}[\sigma]_1^{-1}(1)\otimes[\sigma]_2(1)(b\otimes h)_{(2)}[\sigma]_2^{-1}(1)\\ &= \sum(\sigma_1(1)\otimes 1)(b_{(1)}\otimes h_{(1)})(\sigma_1^{-1}(1)\otimes 1)\otimes(1\otimes\sigma_2(1))\\ &\quad (b_{(2)}\otimes h_{(2)})(1\otimes\sigma_2^{-1}(1))\\ &= \sum(\sigma_1(1)b_{(1)}\sigma_1^{-1}(1)\otimes h_{(1)})\otimes(b_{(2)}\otimes\sigma_2(1)h_{(2)}\sigma_2^{-1}(1)).\end{aligned}$$

$(iii)$ :

$$\begin{aligned}S_{[\sigma]}(b\otimes h) &= \sum S([\sigma]_2(1))[\sigma]_1(1)S(b\otimes h)S^{-1}([\sigma]_2^{-1}(1))[\sigma]_1^{-1}(1)\\ &= \sum S(1\otimes\sigma_2(1))(\sigma_1(1)\otimes 1)S(b\otimes h)S^{-1}(1\otimes\sigma_2^{-1}(1))\\ &\quad (\sigma_1^{-1}(1)\otimes 1)\\ &= \sum(S_B(1)\otimes S_H(\sigma_2(1)))(\sigma_1(1)\otimes 1)(S_B(b)\otimes S_H(h))\\ &\quad (S_B^{-1}(1)\otimes S_H^{-1}(\sigma_2^{-1}(1)))(\sigma_1^{-1}(1)\otimes 1)\end{aligned}$$

$$= \sum (1_B \otimes S_H(\sigma_2(1)))(\sigma_1(1) \otimes 1)(S_B(b) \otimes S_H(h))$$
$$(1_B \otimes S_H^{-1}(\sigma_2^{-1}(1)))(\sigma_1^{-1}(1) \otimes 1)$$
$$= \sum \sigma_1(1) S_B(b) \sigma_1^{-1}(1) \otimes (S_H(\sigma_2(1)) S_H(h)$$
$$S_H^{-1}(\sigma_2^{-1}(1)). \qquad \square$$

EXAMPLE 4. Let $H$ be a finite dimensional Hopf algebra. In Example 3, the coevaluation map $\sigma : k \to H^{op} \otimes H^*$, $\sigma(1) = \sum h_i \otimes h_i^*$ is a skew copairing with inverse $\sigma^{-1}(1) = \sum S^{-1}(h_i) \otimes h_i^*$. Let $A = H^{op} \otimes H^*$. Then the comultiplication on $A_{[\sigma]}$ is $\Delta_{[\sigma]}(h \otimes f) = \sum (h_i h_{(1)} S^{-1}(h_i) \otimes f_{(1)}) \otimes (h_{(2)} \otimes h_i^* f_{(2)} h_i^*)$. The antipode is $S_{[\sigma]}(h \otimes f) = \sum h_i S(h) S^{-1}(h_i) \otimes S(h_i^*) S(f) S^{-1}(h_i^*)$.

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