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CERTAIN GROUPS OF AUTOMORPHISMS OF A UNIVERSAL MINIMAL TRANSFORMATION GROUP

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Abstract. A subgroup $S(X, \gamma)$ of the group of automorphisms of a universal minimal transformation group is introduced and a necessary and sufficient condition for two subgroups to be identical is obtained.

1. Certain groups of automorphisms of a universal minimal transformation group

Throughout this paper, (X, T) will denote the transformation group with compact Hausdorff space X. A closed nonempty subset A of X is called a *minimal subset* if for every $x \in A$, the *orbit* xT is dense in A. A point whose orbit closure is a minimal subset is called an *almost periodic point*. If X is itself minimal, X is called a *minimal set*.

If T is a topological group, a *universal minimal transformation group* for T is a minimal set (M, T) such that every minimal set with group T is a homomorphic image of (M, T). For any group T, a universal minimal transformation group exists and is a unique up to isomorphism ([6]).

A continuous map $\pi : (X, T) \to (Y, T)$ with $\pi(xt) = \pi(x)t$ is called a *homomorphism*. A homomorphism π from (X, T) onto itself is called an *endomorphism* of (X, T), and an isomorphism $\pi : (X, T) \to (X, T)$

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is called an *automorphism* of (X, T). We denote the group of automorphisms of (X, T) by $A(X)$.

The compact Hausdorff space X carries a natural uniformity whose indices are the neighborhood of the diagonal in $X \times X$. Two points x and y of X are called *proximal* provided that for each index $\mathcal{U}$ of X, there exists a $t \in T$ such that $(xt, yt) \in \mathcal{U}$. Two points x and y of X are *regular* if $h(x)$ and y are proximal for some automorphism h of X.

Let $\{(X_i, T) | i \in I\}$ be a family of transformation groups with the same phase group T. The *product transformation group* $(\prod_i X_i, T)$ is defined by the condition that $(x_i | i \in I) \in \prod_i X_i$ and $t \in T$ imply $(x_i | i \in I)t = (x_i t | i \in I)$.

Definition 1.1. ([1]) A minimal transformation group (X, T) is called *regular* if $x, y \in X$, then there is an endomorphism h of (X, T) such that $h(x)$ and y are proximal. Equivalently, if (x, y) is an almost periodic point of $(X \times X, T)$, then there is an endomorphism h of (X, T) such that $h(x) = y$

A minimal set is said to be *coalescent* if every endomorphism is an automorphism. It is well known that regular minimal sets are coalescent, so endomorphism in Definition 1 may be replaced by automorphism.

Let (M, T) be a universal minimal transformation group, which will be fixed from now on, and let G be the group of automorphism of (M, T). Given a minimal transformation group (X, T) and a homomorphism $\gamma : M \to X$, Auslander ([3]) defined a subgroup

$$G(X, \gamma) = \{\alpha \in G \mid \gamma\alpha = \gamma\}.$$

Definition 1.2. ([11]) For a subgroup H of $A(X)$ we define a subgroup $S_H(X, \gamma)$ of G as follows;

$$S_H(X, \gamma) = \{\alpha \in G \mid h\gamma\alpha = \gamma \ \text{ for } \ \text{some } \ h \in H\},$$

where (M,T), (X,T) and $\gamma : M \to X$ are given as in the previous paragraph. If we take $H = \{1_X\}$, then $S_H(X,\gamma)$ coincides with $G(X,\gamma)$. We denote $S_{A(X)}(X,\gamma)$ by $S(X,\gamma)$, simply.

Remark 1. (1) In fact, $S(X,\gamma)$ is a subgroup of G ([11]).

(2) If (X,T) is regular minimal, then $S(X,\gamma)$ is a normal subgroup of G ([11]).

(3) If (X,T) is minimal, then $G(X,\gamma)$ is a normal subgroup of $S(X,\gamma)$ ([12]).

Given a homomorphism $\pi : X \to Y$, the inclusions among the subgroups $G(Y,\pi\gamma)$, $S(X,\gamma)$ and $S(Y,\pi\gamma)$ extend the notions of regular homomorphisms ([12]).

Let $h : X \to X$, $k : Y \to Y$ be the automorphisms of X and Y, respectively. We define $h \times k : X \times Y \to X \times Y$ by $(h \times k)(x,y) = (h(x), k(y))$, $((x,y) \in X \times Y)$. Then $h \times k$ is obviously an automorphism of $X \times Y$. We denote the set of all $h \times k$ by $A(X) \times A(Y)$. It follows that $A(X) \times A(Y) \subset A(X \times Y)$, but $A(X \times Y) \subset A(X) \times A(Y)$ is not true, in general. Regular minimalities of (X,T) and (Y,T) ensure the following lemma.

LEMMA 1.1. *Let (X,T) and (Y,T) be regular minimal transformation groups and let $(X \times X,T)$ and $(Y \times Y,T)$ be minimal. Then $A(X \times Y) = A(X) \times A(Y)$.*

Proof. We need only show that $A(X \times Y) \subset A(X) \times A(Y)$. Let $\varphi \in A(X \times Y)$ and let $\varphi(x,y) = (x',y')$. Since (X,T) and (Y,T) are regular minimal and $(x,x') \in X \times X$, $(y,y') \in Y \times Y$ are almost periodic points, there exist automorphisms $h \in A(X)$ and $k \in A(Y)$ such that $h(x) = x'$ and $k(y) = y'$. Therefore,

$$\varphi(x,y) = (x',y') = (h(x),k(y)) = (h \times k)(x,y),$$

which implies that $\varphi \in A(X) \times A(Y)$. □

Theorem 1.2. *Let $\gamma : M \to X$, $\gamma' : M \to X'$ be homomorphisms from universal minimal transformation group M to minimal sets X and X', respectively. Define $\lambda : M \to X \times X'$ by $\lambda(m) = (\gamma(m), \gamma'(m))$. Then*

(1) $S(X, \gamma) \cap S(X', \gamma') \subset S(X \times X', \lambda)$

(2) Furthermore, if (X, T) and (X', T) are regular minimal and $(X \times X, T)$, $(X' \times X', T)$ are minimal, then $S(X, \gamma) \cap S(X', \gamma') = S(X \times X', \lambda)$.

Proof. (1) Let $\alpha \in S(X, \gamma) \cap S(X', \gamma')$. Then $h\gamma\alpha = \gamma$ and $h'\gamma'\alpha = \gamma'$ for some $h \in A(X)$ and $h' \in A(X')$, that is,

$$h\gamma\alpha(m) = \gamma(m) \text{ and } h'\gamma'\alpha(m) = \gamma'(m)$$

for all $m \in M$. Therefore,

$$\begin{aligned} \lambda(m) &= (\gamma(m), \gamma'(m)) \\ &= (h\gamma\alpha(m), h'\gamma'\alpha(m)) \\ &= (h \times h')(\gamma\alpha(m), \gamma'\alpha(m)) \\ &= (h \times h')(\lambda\alpha(m)), \end{aligned}$$

which shows that $\alpha \in S(X \times X', \lambda)$.

(2) Let (X, T) and (X', T) be regular minimal and let $\alpha \in S(X \times X', \lambda)$. Then there exists a $\varphi \in A(X \times X')$ such that $\varphi\lambda\alpha = \lambda$. From Lemma 1.1, we obtain $\varphi = h \times h'$ for some $h \in A(X)$ and $h' \in A(X')$. It follows that

$$\begin{aligned} (\gamma(m), \gamma'(m)) &= \lambda(m) \\ &= \varphi\lambda\alpha(m) \\ &= (h \times h')(\gamma\alpha(m), \gamma'\alpha(m)) \\ &= (h\gamma\alpha(m), h'\gamma'\alpha(m)) \end{aligned}$$

for all $m \in M$. Therefore,

$$h\gamma\alpha = \gamma \quad \text{and} \quad h'\gamma'\alpha = \gamma',$$

which implies that $\alpha \in S(X,\gamma) \cap S(X',\gamma')$. □

The following theorem is an analogy of Theorem 2 ([3]) by using a subgroup $S_H(X,\gamma)$ instead of $G(X,\gamma)$.

THEOREM 1.3. *Let (X,T) and (X',T) be minimal transformation groups and let $\gamma : M \to X$, $\gamma' : M \to X'$ be homomorphisms. Let H and H' be subgroups of $A(X)$ and $A(X')$, respectively. Suppose that $(X\times X,T)$ and $(X'\times X',T)$ are minimal transformation groups. Then the following are equivalent;*

(1) $S_H(X,\gamma) \subset S_{H'}(X',\gamma')$.
(2) *There are homomorphism $\lambda : M \to X\times X'$ and projections $\pi : X\times X' \to X$, $\pi' : X\times X' \to X'$ with $\gamma = \pi\lambda$, $\gamma' = \pi'\lambda$ and $S_H(X,\pi\lambda) \subset S_{H\times H'}(X\times X',\lambda)$.*

Proof. (1) implies (2). Define $\lambda : M \to X\times X'$ by $\lambda(m) = (\gamma(m),\gamma'(m))$, and let $\pi : X\times X' \to X$ and $\pi' : X\times X' \to X'$ be the coordinate projections to $X\times X'$. Then $\pi\lambda = \gamma$ and $\pi'\lambda = \gamma'$. Now, let $\alpha \in S_H(X,\pi\lambda)$. Then $h\pi\lambda\alpha = \pi\lambda$ for some $h \in H$, that is, $h\gamma\alpha = \gamma$. We also have $h'\gamma'\alpha = \gamma'$ for some $h' \in H'$ by assumption. Then $h\times h' : X\times X' \to X\times X'$ is an automorphism of $X\times X'$ and,

$$\begin{aligned}(h\times h')(\gamma\alpha(m),\gamma'\alpha(m)) &= (h\gamma\alpha(m),h'\gamma'\alpha(m))\\ &= (\gamma(m),\gamma'(m))\\ &= \lambda(m),\end{aligned}$$

for $m \in M$. This show that $(h\times h')\lambda\alpha = \lambda$, that is, $\alpha \in S_{H\times H'}(X\times X',\lambda)$.

(2) implies (1). Let $\alpha \in S_H(X,\gamma)$. Since there exist λ,π,π' with $\gamma = \pi\lambda$, $\gamma' = \pi'\lambda$, $\alpha \in S_H(X,\pi\lambda)$. By hypothesis, $\alpha \in S_{H\times H'}(X\times X',\lambda)$. Therefore, there are $h \in H$ and $h' \in H'$ such that $(h\times h')\lambda\alpha = \lambda$ and we also have $\pi'(h\times h')\lambda\alpha = \pi'\lambda$. It is easy to show that $\pi'(h\times h')\lambda\alpha = h'\pi'\lambda\alpha$. Therefore, we obtain $h'\pi'\lambda\alpha = \pi'(h\times h')\lambda\alpha = \pi'\lambda$, that is, $h'\gamma'\alpha = \gamma'$. This implies that $\alpha \in S_{H'}(X',\gamma')$. □

COROLLARY 1.4. *Let (X,T) and (X',T) be minimal transformation groups and let $\gamma : M \to X$, $\gamma' : M \to X'$ be homomorphisms, and let $(X \times X', T)$ be minimal. Then $G(X,\gamma) \subset G(X',\gamma')$ if and only if there exist homomorphism $\lambda : M \to X \times X'$ and projections $\pi : X \times X' \to X$, $\pi' : X \times X' \to X'$ with $\gamma = \pi\lambda$, $\gamma' = \pi'\lambda$ and $G(X,\pi\lambda) = G(X \times X', \lambda)$.*

Proof. Necessity. Take $H = \{1_X\}$ and $H' = \{1_{X'}\}$, the trivial subgroups of $A(X)$ and $A(X')$, respectively. Then $S_H(X,\gamma) = G(X,\gamma)$ and $S_{H'}(X',\gamma') = G(X',\gamma')$. Then by Theorem 1.3, $S_H(X,\gamma) \subset S_{H'}(X',\gamma)$ implies that there exist λ, π and π' with $\gamma = \pi\lambda$, $\gamma' = \pi'\lambda$ and $G(X,\pi\lambda) \subset G(X \times X', \lambda)$. But $G(X \times X', \lambda) \subset G(X,\pi\lambda)$ is obvious. So, we obtain $G(X,\pi\lambda) = G(X \times X', \lambda)$.

Sufficiency. Let $\alpha \in G(X,\gamma)$. Then $\gamma\alpha = \gamma$. Since $G(X \times X', \lambda) = G(X,\pi\lambda) = G(X,\gamma)$, $\lambda\alpha = \lambda$. So, $\pi'\lambda\alpha = \pi'\lambda$, which shows that $\gamma'\alpha = \gamma'$, that is, $\alpha \in G(X',\gamma')$. □

Let (X,T) and (X',T) be regular minimal and let $(X \times X, T)$ and $(X' \times X', T)$ be minimal. Then $S(X \times X', \lambda) = S(X,\gamma) \cap S(X',\gamma')$ from Theorem 5. Therefore, $S(X \times X', \lambda) = S(X,\gamma)$ if and only if $S(X,\gamma) \subset S(X',\gamma')$.

The following theorem follows immediately from the preceding discussion and Theorem 1.3.

THEOREM 1.5. *Let (X,T) and (X',T) be regular minimal sets and let $(X \times X, T)$ and $(X' \times X', T)$ be minimal. Then*
1) $S(X,\gamma) \subset S(X',\gamma')$ if and only if there are homomorphism $\lambda : M \to X \times X'$ and projections $\pi : X \times X' \to X$, $\pi' : X \times X' \to X'$ with $\gamma = \pi\lambda$, $\gamma' = \pi'\lambda$ and $S(X \times X', \lambda) = S(X,\pi\lambda)$.
2) $S(X,\gamma) = S(X',\gamma')$ if and only if there are homomorphisms λ, π, and π'as in 1), such that $S(X,\pi\lambda) = S(X \times X', \lambda) = S(X',\pi'\lambda)$.

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