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SPLITTING OFF T-SPACES AND DUALITY

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Abstract. We obtain a necessary condition for splitting T-space off a space in terms of cyclic maps, and also obtain a necessary condition for splitting co-T-spaces in terms of cocyclic maps.

1. Introduction

Given a space X, what is the largest torus T^n such that X is homotopy equivalent to $Y \times T^n$ for some space Y?

In [5], Gottlieb answered the above question using the Gottlib group $G_1(X)$. For a space X, the *toral number of* X is defined to be the biggest non negative integer n such that X is homotopy equivalent to $Y \times T^n$ for some space Y. On the other hand, for a subgroup G of $\pi_1(X)$, consider the image of G under the Hurewicz homomorphism h in the homology group. Then $h(G)$ may contain free summands of $H_1(X)$. Then the *Hurewicz rank of* G is defined to be the maximum rank of these free summands. If there is no free summand in $h(G)$ then he said the Hurewicz rank of G is zero and if there is no maximum he say the Hurewicz rank of G is infinite. Then Gottlieb discovered the following beautiful theorem.

Theorem 1.1. ([5] Main Theorem.) *The toral number of X equals to the Hurewicz rank of $G_1(X)$.*

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In [2], Dula and Gottlieb extended the above theorem to a splitting theorem which characterizes when a given space is a cartesian product of an H-space.

THEOREM 1.2. ([2] Theorem 1.3.) *Given spaces X, K and Y the following statements are equivalent;*

(1) *K is an H-space and there exists a space Y such that X is homotopy equivalent to $K \times Y$.*
(2) *There is a class $i : K \to X$ in the generalized Gottlieb set $G(K, X)$ such that $i^{\#} : [X, K] \to [K, K]$ has a right inverse.*
(3) *There are classes i in $G(K, X)$ and $r : X \to K$ such that r is a left homotopy inverse for i.*

On the other hand, it is clear that if G is a topological group then every loop on G can be translated to the base of G and the space of free loop G^{S^1} is homeomorphic to $G \times \Omega G$. More generally, any H-space has this property up to homotopy. In [1], Aguade defined a space X to be a *T-space* if the fibration $\Omega X \to X^{S^1} \to X$ is fiber homotopy equivalent to the trivial fibration $\Omega X \to X \times \Omega X \to X$, where T stands for translation and X^{S^1} is the free loop space of X, ΩX is the based loop space of X. It was also shown [1] that any H-space is a T-space, but the converse does not hold. However, it is not known if there is any finite T-space which fails to be an H-space. In [14], we showed that any H-space is a T-space and any T-space is a G-space using cyclic maps. Also, we defined and studied the concepts of co-T-spaces and some properties of co-T-spaces.

In this paper, we would like to obtain some conditions which characterize when a given space is a cartesian product of a T-space and also to obtain some conditions for dual situation. In Section 2, we can obtain a necessary condition for splitting T-space off a space as follows;

THEOREM 1.3. (Theorem 2.5.) *If K is a T-space and there exists a space Y such that X is homotopy equivalent to $K \times Y$, then there are* *maps $i : K \to X$ and $r : X \to K$ such that r is a left homotopy inverse for i and $ie \in G(\Sigma\Omega K, X)$.*

We do not know whether the converse of the above result holds or not. But we can show that if K is a co-H-space, then the converse of the above result holds. When K is a co-H-space, we know that K is an H-space if and only if K is a T-space. Thus the following corollary is another equivalent conditions for splitting H-space off a space.

Corollary 1.4. (Corollary 2.7.) *Let K be a co-H-space. Then the following conditions are equivalent;*

(1) *K is a T-space and there exists a space Y such that X is homotopy equivalent to $K \times Y$.*
(2) *There is a map $i : K \to X$ such that $i^{\#} : [X, K] \to [K, K]$ has a right inverse and $ie \in G(\Sigma\Omega K, X)$.*
(3) *There are maps $i : K \to X$ and $r : X \to K$ such that r is a left homotopy inverse for i and $ie \in G(\Sigma\Omega K, X)$.*

In Section 3, we would like to study conditions for splitting co-T-space off a space. We show that if X and $K \vee Y$ have the same homotopy type for some space Y, then K is a co-T-space if and only if there exist maps $i : K \to X$ and $r : X \to K$ such that $ri \sim 1$ and $e'r \in DG(X, \Omega\Sigma K)$. Thus we can obtain also a necessary condition for splitting T-space off a space. We can also obtain a necessary and sufficient condition for splitting co-T-space off a space which is a dual result of the above corollary.

Throughout this paper, space means a space of homotopy type of connected locally finite CW complex. We assume also that spaces have non-degenerate base points. All maps shall mean continuous functions. The base point as well as the constant map will be denoted by *. For simplicity, we use the same symbol for a map and its homotopy class. Also, we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$. The identity map of space will be denoted by 1 when it is clear from the context. The diagonal map $\Delta \colon X \to X \times X$ is given by $\Delta(x) = (x, x)$ for each $x \in X$, the folding map $\nabla\colon X \vee X \to X$ is given by $\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$. ΣX denote the reduced suspension of X and ΩX denote the based loop space of X. The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by τ. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$and $\tau(1_{\Sigma X})$, respectively.

2. Splitting T-spaces off a space

This section reviews some results about cyclic maps. In this section, space means a space of homotopy type of connected locally finite CW-complex. A based map $f : A \to X$ is called *cyclic* [13] if there exists a map $F\colon X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$, where $j\colon X \vee A \to X \times A$ is the inclusion and $\nabla\colon X \vee X \to X$ is the folding map. The *Gottlieb set* denoted $G(A, X)$ is the set of all homotopy classes of cyclic maps from A to X. The concept of cyclic maps was first introduced and studied by Gottlieb [3] and Varadarajan [13]. Gottlieb [4] introduced and studied the evaluation subgroups $G_m(X)$ of $\pi_m(X)$. $G_m(X)$ is defined to be the set of all homotopy classes of cyclic maps from S^m to X. A space X satisfying $G_m(X) = \pi_m(X)$ for all m is called a *G-space*.

THEOREM 2.1. [8] *The following are equivalent;*

(1) *X is an H-space.*
(2) *1_X is cyclic.*
(3) *$G(A, X) = [A, X]$ for any space A.*

THEOREM 2.2. [14] *The following are equivalent;*

(1) *X is a T-space.*
(2) *$e : \Sigma\Omega X \to X$ is cyclic.*
(3) *$G(\Sigma A, X) = [\Sigma A, X]$ for any space A.*

PROPOSITION 2.3. [13] *Let $v : A \to X$ be an arbitrary map and let $r : Y \to Z$ have a right homotopy inverse. If $f : X \to Y$ is a cyclic map, then so are $fv : A \to Y$ and $rf : X \to Z$.*

From the above two theorems, we can easily know that any H-space is a T-space. In [11], Stasheff pointed out that ΩX is homotopy commutative if and only if the map $\nabla(e \vee e) : \Sigma\Omega X \vee \Sigma\Omega X \to X$ may be extended to $\Sigma\Omega X \times \Sigma\Omega X$. Thus we know that for any T-space X, ΩX is homotopy commutative. In [7] Hilton showed that X is a co-H-space if and only if $e : \Sigma\Omega X \to X$ has a right homotopy inverse $s : X \to \Sigma\Omega X$. Then we know, from the above theorems and proposition, that if $e : \Sigma\Omega X \to X$ is cyclic and X is a co-H-space, then $1_X \sim es$ is cyclic. Thus we know that H-spaces and T-spaces are equivalent in the category of co-H-spaces. We also obtained the following result.

Proposition 2.4. ([14] Theorem 2.15.) *Let X and $K \times Y$ have the same homotopy type for some space Y. Then K is a T-space if and only if there exist maps $i : K \to X$ and $r : X \to K$ such that $ri \sim 1$ and $ie \in G(\Sigma\Omega K, X)$.*

Thus we have a necessary condition for splitting T-space off a space as follows;

Theorem 2.5. *If K is a T-space and there exists a space Y such that X is homotopy equivalent to $K \times Y$, then there are maps $i : K \to X$ and $r : X \to K$ such that r is a left homotopy inverse for i and $ie \in G(\Sigma\Omega K, X)$.*

We do not know whether the converse of the above result holds or not. But we can show that if K is a co-H-space, then the converse of the above result holds.

Theorem 2.6. *Let K be a co-H-space. If there are maps $i : K \to X$ and $r : X \to K$ such that r is a left homotopy inverse for i and $ie \in G(\Sigma\Omega K, X)$, then K is a T-space and there exists a space Y such that X is homotopy equivalent to $K \times Y$.*

Proof. Let $h : Y \to X$ be the inclusion of the homotopy fiber of $r : X \to K$. By the Milnor's result in [10], ΩK is homotopy equivalent to a CW-complex. Thus in the fiber sequence $\Omega K \to Y \to X$ the base and fiber are homotopy equivalent to a CW-complex, hence by the Stasheff's result in [12], Y is homotopy equivalent to a CW-complex. Since $ie \in G(\Sigma\Omega K, X)$ is cyclic, there is a map $F : \Sigma\Omega K \times X \to X$ such that $Fj \sim \nabla(ie \vee 1)$. Then the composition $rF(1 \times i) : \Sigma\Omega K \times K \to K$ establishes the fact that $e : \Sigma\Omega K \to K$ is cyclic. Thus we know, by Theorem 2.2, that K is a T-space. Since K is a co-H-space, there is a map $s : K \to \Sigma\Omega K$ such that $es \sim 1 : K \to K$. Let $g = F(s \times h) : K \times Y \to X$. Consider the following diagram;

$$\begin{array}{ccccc} Y & \xrightarrow{i_2} & K \times Y & \xrightarrow{p_1} & K \\ \| & & {\scriptstyle g}\downarrow & & \| \\ Y & \xrightarrow{h} & X & \xrightarrow{r} & K \end{array}$$

By the definition of g, the left square commutes, while the right square commutes after π_* is applied. Thus g induces an isomorphism of homotopy groups, and as all spaces are homotopy equivalent to CW-complexes, it follows that g is a homotopy equivalence. □

We can reprove the above theorem as follows; Since K is a co-H-space, there is a map $s : K \to \Sigma\Omega K$ such that $es \sim 1_K$. Since $ie : \Sigma\Omega K \to X$ is cyclic and $es \sim 1_K$, from Proposition 2.3, we have that $i \sim (ie)s \in G(K, X)$. Thus we have, from Theorem 1.2, that K is an H-space and there exists a space Y such that X is homotopy equivalent to $K \times Y$. When K is a co-H-space, we know that K is an H-space if and only if K is a T-space.

Thus the following corollary is another equivalent conditions for splitting H-space off a space.

Corollary 2.7. *Let K be a co-H-space. Then the following conditions are equivalent;*

(1) *K is a T-space and there exists a space Y such that X is homotopy equivalent to $K \times Y$.*

(2) *There is a map* $i : K \to X$ *such that* $i^{\#} : [X, K] \to [K, K]$ *has a right inverse and* $ie \in G(\Sigma\Omega K, X)$.
(3) *There are maps* $i : K \to X$ *and* $r : X \to K$ *such that* r *is a left homotopy inverse for* i *and* $ie \in G(\Sigma\Omega K, X)$.

3. Splitting co-T-spaces off a space

A based map $f\colon X \to A$ is called *cocyclic* [13] if there exists a map $\phi\colon X \to X \vee A$ such that $j\phi \sim (1 \times f)\Delta$, where $j\colon X \vee A \to X \times A$ is the inclusion and $\Delta\colon X \to X \times X$ is the diagonal map. The *dual Gottlieb set* denoted $DG(X, A)$ is the set of all homotopy classes of cocyclic maps from X to A. Haslam [6] introduced and studied the coevaluation subgroups $G^m(X)$ of $H^m(X)$. $G^m(X)$ is defined to be the set of all homotopy classes of cocyclic maps from X to $K(\mathbb{Z}, m)$. A space X satisfying $G^m(X) = H^m(X)$ for all m is called a *G'-space*. In [14], we defined the concept of a co-T-space and showed that any co-H-space is a co-T-space and any co-T-space is a G'-space using cocyclic maps. The purpose of this section is to give necessary sufficient conditions for splitting co-T-space K off a space. The proofs are dual to those of section 2. From now on, every space is assumed to be homotopy equivalent to a connected simply connected CW complex.

Theorem 3.1. [9] *The following are equivalent;*
(1) X *is a co-H-space.*
(2) 1_X *is cocyclic.*
(3) $DG(X, A) = [X, A]$ *for any space* A.

Definition 3.1. A space X is called a *co-T-space* [14] if $e'\colon X \to \Omega\Sigma X$ is cocyclic.

A co-T-space may be characterized by the dual Gottlieb group as follows;

Theorem 3.2. ([14] Theorem 3.3.) *The following are equivalent;*
(1) X *is a co-T-space.*

(2) $e'\colon X \to \Omega\Sigma X$ *is cocyclic.*
(3) $DG(X, \Omega A) = [X, \Omega A]$ *for any space* A.

PROPOSITION 3.3. [13] *Let* $\theta : A \to B$ *be arbitrary map and* $i : Y \to X$ *has a left homotopy inverse. If* $f : X \to A$ *is a cocyclic map, then so are* $\theta f : X \to B$ *and* $fi : Y \to A$.

In [7], Hilton showed that X is an H-space if and only if $e' : X \to \Omega\Sigma X$ has a left homotopy inverse $s' : \Omega\Sigma X \to X$. Then we know, from the above theorems and proposition, that if $e' : X \to \Omega\Sigma X$ is cocyclic, then $1_X \sim s'e'$ is cocyclic. Thus we know that co-H-spaces and co-T-spaces are equivalent in the category of H-spaces. Now we can obtain the following result.

THEOREM 3.4. *Let* X *and* $K \vee Y$ *have the same homotopy type for some space* Y. *Then* K *is a co-*T*-space if and only if there exist maps* $i : K \to X$ *and* $r : X \to K$ *such that* $ri \sim 1$ *and* $e'r \in DG(X, \Omega\Sigma K)$.

Proof. Suppose K is a co-T-space. Since X is homotopy equivalent to $K \vee Y$, there exist maps $f\colon X \to Y \vee K$, $g\colon Y \vee K \to X$ such that $gf \sim 1_X$ and $fg \sim 1_{Y\vee K}$. Let $r = p_2 f : X \xrightarrow{f} Y \vee K \xrightarrow{p_2} K$ and $i = gi_2 : K \xrightarrow{i_2} Y \vee K \xrightarrow{g} X$, where $p_2 : Y \vee K \to K$ is the projection and $i_2 : K \to Y \vee K$ is the inclusion. Then $ri = p_2 fgi_2 \sim p_2 i_2 = 1_X$. Now we show that $e'r : X \to \Omega\Sigma K$ is cocyclic. Since K is a co-T-space,there is a map $\mu : K \to K \vee \Omega\Sigma K$ such that $j\mu = (1 \times e')\Delta$. Consider the composite map $\rho : X \xrightarrow{f} Y \vee K \xrightarrow{(1\vee\mu)} Y \vee K \vee \Omega\Sigma K \xrightarrow{(g\vee 1)} X \vee \Omega\Sigma K$. Then $j\rho \sim (1 \times e'r)\Delta$. Thus $e'r : X \to \Omega\Sigma K$ is cocyclic. On the other hand, suppose there is a map $i : K \to X$ which has a left homotopy inverse $r : X \to K$ and $e'r : X \to \Omega\Sigma K$ is cocyclic. Then we know, from Proposition 3.3, that $e' \sim (e'r)i : K \to \Omega\Sigma K$ is cocyclic. Thus K is a co-T-space. $\square$

THEOREM 3.5. ([2] Theorem 2.2.) *The following conditions are equivalent;*

(1) *K is a co-H-space and there exists a simply connected space Y such that X is homotopy equivalent to $K \vee Y$.*
(2) *There is a map $r : X \to K$ such that $r_\# : [K, X] \to [K, K]$ has a right inverse and $r \in DG(X, K)$.*
(3) *There are maps $i : K \to X$ and $r : X \to K$ such that r is a left homotopy inverse for i and $r \in DG(X, K)$.*

From Theorem 3.4, we have a necessary condition for splitting co-T-space off a space as follows;

Theorem 3.6. *If K is a co-T-space and X is homotopy equivalent to $K \vee Y$, then there exist maps $i : K \to X$ and $r : X \to K$ such that $ri \sim 1$ and $e'r \in DG(X, \Omega\Sigma K)$.*

We do not know whether the converse of the above result holds or not. But we can show that if K is an H-space, the converse of the above result holds.

In the following theorem, X and K are homotopy equivalent to connected and simply connected CW-complexes.

Theorem 3.7. *Let K be an H-space. If there are maps $i : K \to X$ and $r : X \to K$ such that r is a left homotopy inverse for i and $e'r \in DG(X, \Omega\Sigma K)$, then K is a co-T-space and there exists a simply connected space Y such that X is homotopy equivalent to $K \vee Y$.*

Proof. Let $h : X \to Y$ be the inclusion of the homotopy cofiber of $i : K \to X$. It is clear that ΣK and $Y = C_i = M_i/K$ are homotopy equivalent to CW-complexes, where C_i is the mapping cone of $i : K \to X$ and M_i is the mapping cylinder of $i : K \to X$. Since $e'r : X \to \Omega\Sigma K$ is cocyclic, there is a map $\rho : X \to X \vee \Omega\Sigma K$ such that $j\rho = (1 \times e'r)\Delta$. Then the composition $(r \vee 1)\rho i$ establishes the fact that $e' : K \to \Omega\Sigma K$ is cocyclic. Thus we know, by Theorem 3.2, that K is a co-T-space. Since K is an H-space, there is a map $s' : \Omega\Sigma K \to K$ such that $s'e' \sim 1 : K \to K$. Let $g : X \xrightarrow{\rho} X \vee \Omega\Sigma K \xrightarrow{(h \vee s')} Y \vee K$. Consider the following diagram;

$$\begin{array}{ccccc} Y & \xleftarrow{p_1} & Y \vee K & \xleftarrow{i_2} & K \\ \Vert & & \uparrow{\scriptstyle g} & & \Vert \\ Y & \xleftarrow{h} & X & \xleftarrow{i} & K \end{array}$$

By the definition of g, the left square is homotopy commutative, while the right square commutes after H_* is applied. Thus g induces an isomorphism of homology groups, and as all spaces are homotopy equivalent to simply connected CW-complexes, it follows that g is a homotopy equivalence. □

We can also reprove the above theorem as follows; Since K is an H-space, there is a map there is a map $s' : \Omega\Sigma K \to K$ such that $s'e' \sim 1 : K \to K$. Since $e'r : X \to \Omega\Sigma K$ is cocyclic and $s'e' \sim 1_K$, from Proposition 3.3, we have that $r \sim s'(e'r) \in DG(X, K)$. Thus we have, from Theorem 3.5, that K is a co-H-space and there exists a simply connected space Y such that X is homotopy equivalent to $K \vee Y$. Since K is an H-space, we know that K is a co-H-space if and only if K is a co-T-space.

Thus the following corollary is another equivalent conditions for splitting co-H-space off a space.

Corollary 3.8. *Let K be an H-space. Then the following conditions are equivalent;*

(1) *K is a co-T-space and there exists a simply connected space Y such that X is homotopy equivalent to $K \vee Y$.*
(2) *There is a map $r : X \to K$ such that $r_\# : [K, X] \to [K, K]$ has a right inverse and $e'r \in DG(X, \Omega\Sigma K)$.*
(3) *There are maps $i : K \to X$ and $r : X \to K$ such that r is a left homotopy inverse for i and $e'r \in DG(X, \Omega\Sigma K)$.*

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