JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **16**, No.1, June 2003

SOME RESULTS RELATED TO EXTREMAL LENGTH, *II*

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Abstract. In this note, we introduce the concept of the extremal length of a curve family in the complex plane and apply the extremal length to the boundary behavior of analytic functions. We consider some geometric applications of extremal length and establish applications connected with the logarithmic capacity.

1. Introduction

The method of extremal length is a useful tool in a wide variety of areas. Especially, it has been successfully applied to conformal mappings, analytic functions of a complex variable. Extremal length was introduced as a conformally invariant measure of curve families. This development appeared in Ahlfors and Beurling [2].

A set which is of capacity zero is of linear measure zero. Most of the concepts and the mathematical methods in analysis would be of little practical interest-think of considering the set of measure zero. But, by using the method of extremal length, we can easily see that some results are in connection with the set of measure zero.

Throughout this paper, C will denote the finite complex plane, D is a domain (open and connected set) in C, g is an arbitrary function defined on D, ∂D is the boundary of D and $\mathrm{cl}(D)$ is the closure of D.

*This research was supported by the Soonchunhyang University Basic Science Research Fund 2002.

Received by the editors on May 24, 2003.

2000 *Mathematics Subject Classifications* : Primary 30C62, 30C85.

Key words and phrases: extremal length, logarithmic capacity, cluster set.

2. Extremal length

Let Γ be a family whose elements γ are locally rectifiable curves in D. We shall introduce a geometric quantity $\lambda(\Gamma)$, called the extremal length of Γ. Let $\rho(z)$ be a non-negative Borel measurable function defined on C. Every curve γ has a well-defined

$$L(\gamma,\, \rho) = \int_\gamma \rho(z)\, |dz|\,, \qquad z = x + iy \tag{1}$$

which may be infinite, and D has a

$$A(D,\, \rho) = \iint_D \rho(z)^2\, dxdy\,. \tag{2}$$

In order to define an invariant which depends on the whole set Γ, we introduce

$$L(\Gamma,\, \rho) = \inf_{\gamma\in\Gamma} L(\gamma,\, \rho), \tag{3}$$

where we agree that $L(\Gamma,\, \rho) = \infty$ in the case Γ is empty.

To obtain a quantity that does not change when the weight function ρ is multiplied by a constant we form the homogeneous expression $[L(\Gamma,\, \rho)]^2/A(D,\, \rho)$.

Definition 2.1. ([1]) The extremal length of Γ in D is defined as

$$\lambda_D(\Gamma) = \sup_\rho \frac{[L(\Gamma,\, \rho)]^2}{A(D,\, \rho)}\,, \tag{4}$$

where ρ is subject to the condition $0 < A(D,\, \rho) < \infty$, obviously $0 \leq \lambda(\Gamma) \leq \infty$.

Remark 1. 1. ([1]) $\lambda_D(\Gamma)$ depends only on Γ and not on D. Accordingly, we shall henceforth simplify the notation to $\lambda(\Gamma)$.
2. ([14]) Since almost every curve in C is rectifiable, the non-rectifiable curves of a family Γ have no influence on the extremal length of Γ. Accordingly, let us use curve or arc instead of locally rectifiable curve.

3. $\rho(z)$ will be called allowable if it satisfies the condition (2).

There are two special cases in which the extremal length is very easy to determine explicitly.

PROPOSITION 2.1. *Let R be a rectangle of sides a and b. Let Γ be the family of curves in R which join the sides of length a. Then*

$$\lambda(\Gamma) = \frac{b}{a}\,.$$

Proof. See Example 4.2 in [1]. □

PROPOSITION 2.2. *Let S be the annulus $S = \{z \mid r_1 < |z| < r_2\}$ and Γ the family of arcs in S which join the two contours. Then*

$$\lambda(\Gamma) = \frac{1}{2\pi}\log\left(\frac{r_2}{r_1}\right)\,.$$

Proof. For any $\rho(z)$, we have

$$\int_{r_1}^{r_2}\rho\,dr \le L(\Gamma,\,\rho), \qquad \iint_S \rho\,drd\theta \ge 2\pi L(\Gamma,\,\rho)$$

Then, by the Schwarz inequality (ref. [7]),

$$\begin{aligned} 4\pi^2[L(\Gamma,\,\rho)]^2 &\le \left(\iint_S \rho\,drd\theta\right)^2 \\ &\le \left(\iint_S \rho^2\frac{1}{r}\,drd\theta\right)\left(\iint_S r\,drd\theta\right) \\ &= 2\pi\log\frac{r_2}{r_1}\iint_S \rho^2 r\,drd\theta\,. \end{aligned}$$

This proves $\lambda(\Gamma) \le \dfrac{1}{2\pi}\log\dfrac{r_2}{r_1}$.

For $\rho = \dfrac{1}{r}$, we have

$$L\left(\Gamma,\,\frac{1}{r}\right) = \log\frac{r_2}{r_1}, \qquad A\left(S,\,\frac{1}{r}\right) = 2\pi\log\frac{r_2}{r_1}\,.$$

Therefore $\lambda(\Gamma) \geq \frac{1}{2\pi} \log \frac{r_2}{r_1}$.

This completes the proof. □

As immediate consequences of Proposition 2.1, we have the following corollary.

COROLLARY 2.3. *Let R be as in Proposition 2.1. Let $\Gamma^{'}$ be the family of curves in R which join the sides of length b. Then*

$$\lambda(\Gamma) \cdot \lambda(\Gamma^{'}) = 1\,.$$

DEFINITION 2.2. We say that $\lambda(\Gamma^{'})$ is the conjugate extremal length of $\lambda(\Gamma)$.

The notion of extremal length may be justified somewhat by the following properties of extremal length which are frequently used in our paper.

THEOREM 2.4 (Conformal invariance of extremal length). *Let $Z^* = f(z)$ be a $1-1$ conformal mapping on D upon a domain D^* and Γ be a family of curves in D. then*

$$\lambda(\Gamma) = \lambda[f(\Gamma)]\,.$$

Proof. See [11]. □

THEOREM 2.5 (Comparison principle of extremal length). *For two curve families Γ_1, Γ_2, if every $\gamma_2 \in \Gamma_2$ contains a $\gamma_1 \in \Gamma_1$, then*

$$\lambda(\Gamma_1) \leq \lambda(\Gamma_2)\,.$$

Proof. See Theorem 4.1 in [1]. □

Remark 2. 1. ([1]) Briefly, the set Γ_2 of fewer or longer curves has the larger extremal length.

2. ([12]) There is a physical interpretation of extremal length. Think of the curve family Γ as representing a system of homogeneous electric wires. Then the extremal length $\lambda(\Gamma)$ represents the resistance of Γ. The above Theorem 2.5 reflect the fact that systems

of fewer or longer wires have greater resistance (smaller conductance).

3. Geometric applications

The simplest example concerns the general quadrilateral (a Jordan curve with four distinguished points).

THEOREM 3.1. *Let Q be a general quadrilateral of area M. Let a be the length of the shortest arc in Q connecting one pair of opposite sides. Let b be the length of the shortest arc in Q connecting the other pair of sides. Then*

$$a \cdot b \leq M .$$

Proof. A purely geometric proof of this theorem was given by A. Besicovitch [5]. The geometric proof is difficult. But the use of extremal length makes the proof trivial.

Let Γ be the family of arcs in Q which join one pair of opposite sides, and Γ' the family of arcs in Q which join the other two sides. There is a $1-1$ conformal mapping f of Q onto some rectangle such that $f(\Gamma)$ and $f(\Gamma')$ are the families of arcs which join opposite sides of the rectangle. By Corollary 2.3,

$$\lambda[f(\Gamma)] \cdot \lambda[f(\Gamma')] = 1 .$$

And by Theorem 2.4,

$$\lambda(\Gamma) \cdot \lambda(\Gamma') = 1 .$$

Take non-negative Borel measurable function $\rho = 1$. Then we have

$$\frac{a^2}{M} \cdot \frac{b^2}{M} = \frac{[L(\Gamma, 1)]^2}{A(Q, 1)} \cdot \frac{[L(\Gamma', 1)]^2}{A(Q, 1)} \leq \lambda(\Gamma) \cdot \lambda(\Gamma') = 1 ,$$

and the Theorem follows immediately. □

THEOREM 3.2. *Let B be a ring domain and let B_0 and B_1 denote the bounded component and unbounded component of the complement B^c of B, respectively. Let ∂B_0 and ∂B_1 denote the two components* *of the boundary of B. Let a be the length of the shortest arc in B connecting ∂B_0 and ∂B_1 and b the length of the Jordan curve, ∂B_0. Then*

$$a \cdot b \leq M\,,$$

where M is the area of B.

In our discussion we will need the following. We say the closed curve γ in B separates B_0 and B_1 if γ has non-zero winding number about the points of B_0.

LEMMA 3.3. *Let B, B_0, B_1, ∂B_0, and ∂B_1 be as in Theorem 3.2. Let Γ_B be the family of all curves in B connecting ∂B_0 and ∂B_1. And let ${\Gamma_B}'$ be the family of all closed curves in B which separate B_0, B_1. Then*

$$\lambda(\Gamma_B) \cdot \lambda({\Gamma_B}') = 1\,.$$

We say that $\lambda({\Gamma_B}')$ is the conjugate extremal length of $\lambda(\Gamma_B)$ (cf. Definition 2.2).

Proof of Theorem 3.2. Using Lemma 3.3 and the similar argument as in the proof of Theorem 3.1, we can show the above. □

4. Logarithmic capacity and Extremal length

DEFINITION 4.1. ([3]) Let E be a bounded Borel set in C and μ a non-negative completely additive set function defined for the Borel subsets of E. The measure μ is then called a positive mass-distribution on E. Let μ be a positive mass-distribution on E with total mass unity. Then

$$U^{\mu}(z) = \int_E \log\left|\frac{1}{z-\zeta}\right| \, d\mu(\zeta)$$

is called a logarithmic potential of μ on E. Let

$$V_\mu(E) = \sup_{z\in E} U^{\mu}(z), \qquad V = \inf_{\mu} V_\mu(E),$$

where μ ranges over all mass-distribution on E with total mass unity.

We define the logarithmic capacity(simply, capacity), Cap(E) of E by $\mathrm{Cap}(E) = \exp(-V)$.

The following cantor set has positive capacity and linear measure 0.

EXAMPLE 4.1. *For the Cantor ternary set* $E\left(\left\{\frac{2}{3}\right\}\right)$, *Cap*$(E) \geq \frac{1}{18}$.

Proof. See [3]. □

Early in the development of extremal length, Ahlfors and Beurling related it to logarithmic capacity. The double role of capacity as a conformal invariant and a geometric quantity permits us to gain relevant information about analytic functions. For the convenience of the reader we shall recall some basic properties.

PROPOSITION 4.2. *(*[7]*)*

1. $0 \leq \mathit{Cap}(E) < \infty$.
2. $E_1 \subset E_2$ *implies* $\mathit{Cap}(E_1) \leq \mathit{Cap}(E_2)$.
3. *A set which is of capacity zero is of liner measure zero.*
4. *The capacity of a countable set is also zero.*
5. *The union of a countable set of sets of capacity zero is of capacity zero.*

5. Some results

DEFINITION 5.1. ([10]) If every component of a set is a point, the set is called totally disconnected.

For example, $\left\{\frac{1}{n} \mid n \in N\right\} \cup \{0\}$ is totally disconnected compact set.

DEFINITION 5.2. ([4]) Let γ be a curve at $z_0 \in \mathrm{cl}(D)$, then the cluster set of g at z_0 along γ, denoted by $C_\gamma(g, z_0)$, is defined to be the set of all points ω with the property that, for some sequence of points $\{z_n\}$ on γ for which

$$\lim_{n\to\infty} z_n = z_0, \text{ we have } \lim_{n\to\infty} g(z_n) = \omega\,.$$

The value ω is called a *cluster value* of g at z_0 along γ.

The following theorem applies the extremal length to the analytic function defined on the domain with a number of holes.

THEOREM 5.1. *Let $f(z)$ be a bounded single-valued analytic function in the complement of E, Where E is a totally disconnected compact set of positive capacity in C. Then it is not the case that for each z in E, except for those z in a set of capacity zero, there exist two curves at z in the complement of E on which $f(z)$ has the cluster values ω_1 and ω_2 $(\omega_1 \neq \omega_2)$.*

Proof. See Theorem 3.5. in [6]. □

We obtain the following corollary in connection with the ambiguous points.

DEFINITION 5.3. ([4]) We say that a point $z_0 \in \partial D$ is an *ambiguous point* of g, provided that there exist curves γ_1 and γ_2 at z_0 such that

$$C_{\gamma_1}(g,\, z_0) \cap C_{\gamma_2}(g,\, z_0) = \emptyset\,.$$

COROLLARY 5.2. *Let E and $f(z)$ be as in Theorem 5.1. Then it is not the case that, except for those z in a set of capacity zero, each z in E is an ambiguous points of $f(z)$.*

Proof. Assume that the statement is not true, that is, suppose that for each z in E, there exist two curves γ_1 and γ_2 in E^c at z such that

$$C_{\gamma_1}(f,\, z) \cap C_{\gamma_2}(f,\, z) = \emptyset\,.$$

Since $C_{\gamma_1}(f,\, z)$ and $C_{\gamma_2}(f,\, z)$ are disjoint compact sets (see [4]), there exist disjoint open sets G_1 and G_2 such that

$$C_{\gamma_1}(f,\, z) \subset G_1, \qquad C_{\gamma_2}(f,\, z) \subset G_2\,.$$

Hereafter, the proof of this corollary is very similar to the proof of Theorem 5.1. The corollary is established. □

We consider the applications of extremal length to conformal mappings. A purely function-theoretic proof of following theorem is difficult. The use of extremal length, however, makes the proof trivial.

THEOREM 5.3. *Let D be a Jordan domain, let z_0, z_1 be two distinct points of ∂D, and denote by C_1 and C_2 the two curves between z_0 and z_1, Where C_1, $C_2 \subset \partial D$. Let $f(z)$ be a $1-1$ conformal mapping defined on D satisfying*

$$\iint_D \left| f^{'}(z) \right|^2 dxdy < \infty .$$

If $f(z)$ has the cluster values ω_1 and ω_2 (ω_1, $\omega_2 \neq \infty$) for some sequence of points $\{z_n\}$ on C_1 and C_2 converging to z_0 respectively, then

$$\omega_1 = \omega_2 .$$

For our proof we will need the following.

DEFINITION 5.4. ([4]) Let D be a simply connected domain. A crosscut of D is a Jordan curve γ in D which in both directions tends to a boundary point.

LEMMA 5.4. *Let B, B_0, B_1, ∂B_0 and ∂B_1 be as in Theorem 3.2. Let Γ_B be as in Lemma 3.3. Then*

$$\lambda(\Gamma_B) = \infty$$

iff B_0 consists of a single point.

Proof. See [13]. □

Proof of Theorem 5.3. Let Γ be the family of all crosscuts γ in B_D connecting points of C_1 and points of C_2, where

$$B_D = \{z \mid 0 < |z - z_0| < r_0\} \cap D .$$

Here r_0 is sufficiently small positive. Then

$$|\omega_1 - \omega_2| \leq \inf_{\gamma \in \Gamma} \int_\gamma |f'(z)| \, |dz| .$$

There remains to show that

$$\inf_{\gamma\in\Gamma}\int_{\gamma}|f'(z)|\,|dz| = 0\,. \tag{5}$$

Since

$$B = \{z \mid 0 < |z - z_0| < r_0\}$$

is a ring domain, by the Lemma 5.4, we see that

$$\lambda(\Gamma_B) = \infty,$$

where, Γ_B is as in Lemma 5.4.

Suppose now that we are considering $\Gamma_B^{'}$ the family of all simple closed curves in B separating z_0 from $\{z \mid |z - z_0| = r_0\}$, then $\lambda(\Gamma_B^{'})$ is the conjugate extremal length of $\lambda(\Gamma_B)$. Hence by the Lemma 3.3,

$$\lambda(\Gamma_B^{'}) = 0\,.$$

And clearly,

$$\Gamma_B^{'} \subset \Gamma$$

Thus by Theorem 2.4, we see that

$$\lambda(\Gamma) = 0\,. \tag{6}$$

On the other hand, we choose the allowable function $\rho(z) = |f'(z)|$ on D, then

$$\iint_D [\rho(z)]^2\,dxdy < \infty\,. \tag{7}$$

Hence by (6), (7), we have (5).

This completes the proof of the theorem. □

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