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CONTINUITY OF LINEAR OPERATOR INTERTWINING WITH DECOMPOSABLE OPERATORS AND PURE HYPONORMAL OPERATORS

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Abstract. In this paper, we show that for a pure hyponormal operator the analytic spectral subspace and the algebraic spectral subspace are coincide. Using this result, we have the following result: Let T be a decomposable operator on a Banach space X and let S be a pure hyponormal operator on a Hilbert space H. Then every linear operator $\theta : X \to H$ with $S\theta = \theta T$ is automatically continuous.

1. Introduction

Let X and Y be Banach spaces and consider a linear operator $\theta : X \to Y$. The basic automatic continuity problem is to derive the continuity of θ from some prescribed algebraic conditions. For example, if $\theta : X \to Y$ is a linear operator intertwining with $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, that is $\theta T = S\theta$, one may look for algebraic conditions on T and S which force θ to be continuous.

The study of continuity of a linear operator θ intertwining with T and S was initiated by Johnson and Sinclair [3]. In [3] necessary conditions on T and S for the continuity of θ were obtained for the operator S with countable spectrum.

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In 1973 Vrbová presented an automatic continuity result concerning an intertwining operator with operators having suitable spectral decomposition properties [11].

In 1986 Laursen and Neumann introduced super-decomposable operators in [6] in order to consider necessary conditions for automatic continuity of intertwining operators: this class of operators contains most of interesting examples of decomposable operators. Since [6], the study of automatic continuity of intertwining linear operators has been closely related to the classification of decomposable operators.

In this paper, we show that for a pure hyponormal operator the analytic spectral subspace and the algebraic spectral subspace are coincide. Using this result, we have the following result: Let T be a decomposable operator on a Banach space X and let S be a pure hyponormal operator on a Hilbert space H. Then every linear operator $\theta : X \to H$ with $S\theta = \theta T$ is automatically continuous.

2. Preliminaries

Throughout this paper we shall use the standard notions and some basic results on the theory of decomposable operators and automatic continuity theory. Let X be a Banach space over the complex plane $\mathbb{C}$. And let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on a Banach space X. Given an operator $T \in \mathcal{L}(X)$, $\mathrm{Lat}(T)$ denotes the collection of all closed T-invariant linear subspaces of X, and for an $Y \in \mathrm{Lat}(T)$ $T|Y$ denotes the restriction of T on Y.

Definition 1. Let $T : X \to X$ be a linear operator on a Banach space X. Let F be a subset of the complex plane $\mathbb{C}$. Consider the class of all linear subspaces Y of X which satisfy $(T - \lambda)Y = Y$ for all $\lambda \notin F$ and let $E_T(F)$ denote the span of all such subspaces Y of X. $E_T(F)$ is called an *algebraic spectral subspace* of T.

In the next Remark, we collect a number of results on algebraic spectral subspaces. These results are found in [7].

REMARK 2. (1) It is clear that $(T-\lambda)E_T(F) = E_T(F)$ for all $\lambda \notin F$ as well so that it is the largest linear subspace with this property.

(2) By the definition of the algebraic spectral subspace, it is clear that

$$E_T(F_1) \subseteq E_T(F_2) \quad \text{for} \quad F_1 \subseteq F_2.$$

(3) Let A be a linear operator on a vector space X with $AT = TA$. For a given subset F of $\mathbb{C}$ and $\lambda \notin F$, we obtain

$$(T-\lambda)AE_T(F) = A(T-\lambda)E_T(F) = AE_T(F).$$

By the maximality of $E_T(F)$ we have

$$AE_T(F) \subseteq E_T(F).$$

That is, the space $E_T(F)$ is a hyper-invariant subspace of T.

(4) It is well known that if $\{F_\alpha\}$ is a family of subsets of $\mathbb{C}$, then

$$E_T(\bigcap_\alpha F_\alpha) = \bigcap_\alpha E_T(F_\alpha).$$

A linear subspace Z of X is called a *T-divisible subspace* if

$$(T-\lambda)Z = Z \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

Hence $E_T(\emptyset)$ is precisely the largest T-divisible subspace. There is an operator which has non-trivial divisible subspaces. Indeed, the Volterra operator has a non-trivial divisible subspace.

PROPOSITION 3. *Let $T \in \mathcal{L}(X)$. If Z is a closed T-divisible subspace of X, then $Z = \{0\}$.*

Proof. Let $\lambda \in \partial\sigma(T|Z)$, where $\partial\sigma(T|Z)$ denotes the boundary of the spectrum of T restricted to Z. By the general theory of Banach algebras, there exists a sequence $\langle T_n \rangle$ of linear operators in Z with $\|T_n\| = 1$ for all $n \in \mathbb{N}$ and $T_n(T-\lambda) \to 0$. Since $(T-\lambda)Z = Z$, by the open mapping theorem, $kB \subseteq (T-\lambda)B$ for some $k > 0$ where B is the unit ball in Z. For sufficiently large n with $\|T_n(T-\lambda)\| < \frac{k}{2}$, we have

$$kT_nB \subseteq T_n(T-\lambda)B \subseteq \frac{k}{2}B.$$

Then $T_nB \subseteq \frac{1}{2}B$, which contradicts the assumption that $\|T_n\| = 1$. □

For a given $T \in \mathcal{L}(X)$, let $\sigma(T)$ and $\rho(T)$ denote the spectrum and the resolvent set of T, respectively. The *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ which satisfies

$$(T-\lambda)f(\lambda) = x \quad \text{for all} \quad \lambda \in U.$$

The *local spectrum* $\sigma_T(x)$ of T at x is then defined as

$$\sigma_T(x) = \mathbb{C} \setminus \rho_T(x).$$

Clearly, the local resolvent set $\rho_T(x)$ is open, and the local spectrum $\sigma_T(x)$ is closed. For each $x \in X$, the function $f(\lambda) : \rho(T) \to X$ defined by $f(\lambda) = (T-\lambda)^{-1}x$ is analytic on $\rho(T)$ and satisfies

$$(T-\lambda)f(\lambda) = x \quad \text{for all} \quad \lambda \in \rho(T).$$

Hence the resolvent set $\rho(T)$ is always subset of $\rho_T(x)$ and hence $\sigma_T(x)$ is always subset of $\sigma(T)$. The analytic solutions occurring in the definition of the local resolvent set may be thought of as local extensions of the function $(T-\lambda)^{-1}x$. There is no uniqueness implied. If for each $x \in X$ there is the unique analytic extension of $(T-\lambda)^{-1}x$, then T is said to have the *single-valued extension property*, abbreviated

SVEP. Hence if T has the SVEP, then there is the maximal analytic extention of $(T-\lambda)^{-1}x$ from $\rho(T)$ to $\rho_T(x)$.

Given an arbitrary operator $T \in L(X)$ and for any set $F \subseteq \mathbb{C}$, we define the *analytic spectral subspace* of T by

$$X_T(F) = \{x \in X \mid \sigma_T(x) \subseteq F\}.$$

In the next Remark, we collect a number of results on analytic spectral subspaces. These results are found in [7].

REMARK 4 (1) By the definition of the analytic spectral subspace, it is clear that

$$X_T(F_1) \subseteq X_T(F_2) \quad \text{for} \quad F_1 \subseteq F_2.$$

(2) It is well known that the space $X_T(F)$ is a hyper-invariant subspace of T.

(3) It is easy to see that

$$X_T(F) = X_T(F \cap \sigma(T)).$$

(4) For all $\lambda \in \mathbb{C} \setminus F$,

$$(T-\lambda)X_T(F) = X_T(F)$$

This implies that

$$X_T(F) \subseteq E_T(F) \quad \text{for all} \quad F \subseteq \mathbb{C}.$$

(5) If $\{F_\alpha\}$ is a family of subsets of $\mathbb{C}$, then

$$X_T(\bigcap_\alpha F_\alpha) = \bigcap_\alpha X_T(F_\alpha).$$

(6) It is well known that T has the SVEP if and only if $X_T(\emptyset) = \{0\}$.

An operator $T \in \mathcal{L}(X)$ is called *decomposable* if, for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, there exist $Y, Z \in \mathrm{Lat}(T)$ such that

$$\sigma(T|Y) \subseteq U, \ \ \sigma(T|Z) \subseteq V \ \text{ and } \ Y + Z = X.$$

Decomposable operators are rich. For example, normal operators, spectral operators in the sense of Dunford, operators with totally disconnected spectrums and hence compact operators are decomposable.

Let $\mathcal{F}(\mathbb{C})$ denote the family of all closed subsets of $\mathbb{C}$ and let $\mathcal{S}(X)$ denote the family of all closed linear subspaces of X.

DEFINITION 5. (1) A map $\mathcal{E}(\cdot) : \mathcal{F}(\mathbb{C}) \to \mathcal{S}(X)$ is called *stable* if it satisfies the following two conditions:

(i) $\mathcal{E}(\emptyset) = \{0\}$, $\mathcal{E}(\mathbb{C}) = X$.

(ii) $\mathcal{E}(\bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty} \mathcal{E}(F_n)$ for any sequence $\{F_n\}$ in $\mathcal{F}(\mathbb{C})$.

(2) A map $\mathcal{E}(\cdot) : \mathcal{F}(\mathbb{C}) \to \mathcal{S}(X)$ is called a *spectral capacity* if $\mathcal{E}(\cdot)$ is stable and satisfies the following condition:

(iii) $X = \sum_j \mathcal{E}(\overline{G}_j)$ for every finite open cover $\{G_j\}$ of $\mathbb{C}$.

We say that $\mathcal{E}(\cdot)$ is *order preserving* if it preserves the inclusion order. Clearly a stable map is order preserving. It is well known that T is decomposable if and only if there exists a spectral capacity $\mathcal{E}(\cdot)$ such that $\mathcal{E}(F) \in \mathrm{Lat}(T)$ and $\sigma(T|\mathcal{E}(F)) \subseteq F$ for each closed set $F \subseteq \mathbb{C}$. In this case the spectral capacity of a closed subset F of $\mathbb{C}$ is uniquely determined and it is the analytic spectral subspace $X_T(F)$.

Let θ be a linear operator from a Banach space X into a Banach space Y. The space

$$\mathfrak{S}(\theta) = \{y \in Y : \text{ there is a sequence } x_n \to 0 \text{ in } X \text{ and } \theta x_n \to y\}$$

is called the *separating space* of θ. It is easy to see that $\mathfrak{S}(\theta)$ is a closed linear subspace of Y. By the closed graph theorem, θ is continuous if and only if $\mathfrak{S}(\theta) = \{0\}$. The following lemma is found in [10].

LEMMA 6. *Let* X *and* Y *be Banach spaces. If* R *is a continuous linear operator from* Y *to a Banach space* Z, *and if* $\theta : X \to Y$ *is a linear operator, then* $(R\mathfrak{S}(\theta))^- = \mathfrak{S}(R\theta)$. *In particular,* $R\theta$ *is continuous if and only if* $R\mathfrak{S}(\theta) = \{0\}$.

The next lemma states that a certain descending sequence of separating space which obtained from θ via a countable family of continuous linear operators is eventually constant. This lemma is proved in [3], [4] and [10].

STABILITY LEMMA. *Let* $\theta : X_0 \to Y$ *be a linear operator between the Banach spaces* X_0 *and* Y *with separating space* $\mathfrak{S}(\theta)$, *and let* $\langle X_i : i = 1, 2, \dots \rangle$ *be a sequence of Banach spaces. If each* $T_i : X_i \to X_{i-1}$ *is continuous linear operator for* $i = 1, 2, \dots$, *then there is an* $n_0 \in \mathbb{N}$ *for which*

$$\mathfrak{S}(\theta T_1 T_2 \dots T_n) = \mathfrak{S}(\theta T_1 T_2 \dots T_{n_0}) \quad \text{for all } n \geq n_0.$$

The following lemma, known as *localization of the singularities*, is adopted from [5].

LEMMA 7. *Let* X *and* Y *be Banach spaces. Suppose that* $\mathcal{E}_X : \mathcal{F}(\mathbb{C}) \to \mathcal{S}(X)$ *is an order preserving map such that* $X = \mathcal{E}_X(\overline{U}) + \mathcal{E}_X(\overline{V})$ *whenever* $\{U, V\}$ *is an open cover of* $\mathbb{C}$. *And suppose that* $\mathcal{E}_Y : \mathcal{F}(\mathbb{C}) \to \mathcal{S}(Y)$ *is a stable map. If* $\theta : X \to Y$ *is a linear operator for which*

$$\mathfrak{S}(\theta | \mathcal{E}_X(F)) \subseteq \mathcal{E}_Y(F) \quad \text{for every } F \in \mathcal{F}(\mathbb{C}),$$

then there is a finite set $\Lambda \subseteq \mathbb{C}$ *for which* $\mathfrak{S}(\theta) \subseteq \mathcal{E}_Y(\Lambda)$.

This lemma tells us that under appropriate assumptions on a linear operator which have a large lattice of closed invariant subspaces the separating space will be contained eventually in a small closed invariant subspace.

We need the next theorem, known as Mittag-Leffler Theorem of Bourbaki, which is found in [10].

Mittag-Leffler Theorem. *Let* $\langle X_n : n = 0, 1, 2, \dots \rangle$ *be a sequence of complete metric spaces, and for* $n = 1, 2, \dots$, *let* $f_n : X_n \to X_{n-1}$ *be a continuous map with* $f_n(X_n)$ *dense in* X_{n-1}. *Let* $g_n = f_1 \circ \cdots \circ f_n$. *Then* $\bigcap_{n=1}^{\infty} g_n(X_n)$ *is dense in* X_0.

3. Continuity of linear operator intertwining with decomposable operators and pure hyponormal operators

Let H be a Hilbert space over the complex plane $\mathbb{C}$ with the inner product $(\cdot, \cdot)$ and let $\mathcal{L}(H)$ denote the Banach algebra of bounded linear operators on H. An operator $T \in \mathcal{L}(H)$ is said to be *hyponormal* if its self commutator $[T^*, T] = T^*T - TT^*$ is positive, that is

$$((T^*T - TT^*)\xi, \xi) \geq 0,$$

or equivalently,

$$\|T^*\xi\| \leq \|T\xi\|$$

for every $\xi \in H$.

It is well known that every hyponormal operator T has the single valued extension property and for any closed set F in $\mathbb{C}$, the analytic spectral subspace $H_T(F)$ is closed [7].

Definition 8. A hyponormal operator T on a Hilbert space H is said to be *pure hyponormal* if there is no nontrivial normal operator which is a direct orthogonal summand of T.

The following proposition is found in [8].

PROPOSITION 9. *Let T be a pure hyponormal operator on a Hilbert space H. Then*

$$H_T(F) = \bigcap_{\lambda \notin F} (T-\lambda)H,$$

for any closed subset $F \subseteq \mathbb{C}$.

The following proposition is found in [2].

PROPOSITION 10. *Let T be a pure hyponormal operator on a Hilbert space H. Then T has no eigenvalues.*

For a pure hyponormal operator, the following proposition allows us to combine the analytic tools associated with the space $H_T(F)$ with the algebraic tools associated with the space $E_T(F)$.

PROPOSITION 11. *Let T be a pure hyponormal operator on a Hilbert space H. Then for any closed set F of $\mathbb{C}$, $H_T(F) = E_T(F)$.*

Proof. Let F be a closed subset of $\mathbb{C}$. From the definition of the algebraic spectral subspace, it is clear that

$$E_T(F) \subseteq \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T-\lambda)^n H.$$

By Proposition 9, we have

$$E_T(F) \subseteq \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T-\lambda)^n H \subseteq \bigcap_{\lambda \notin F} (T-\lambda)H = H_T(F).$$

Therefore we have,

$$H_T(F) = E_T(F)$$

for any closed subset F of $\mathbb{C}$. □

By the above proposition, pure hyponormal operators do not have non-trivial divisible subspaces.

Let T and S be bounded linear operators on Banach spaces X and Y, respectively. A linear operator $\theta : X \to Y$ is said to be an *intertwining linear operator* with T and S if $S\theta = \theta T$.

PROPOSITION 12. *Assume that $T \in \mathcal{L}(X)$ has the single-valued extension property and that a pure hyponormal operator $S \in \mathcal{L}(H)$ Then every linear transformation $\theta : X \to H$ with the property $S\theta = \theta T$ necessarily satisfies the following:*

$$\theta X_T(F) \subseteq H_S(F) \quad \textit{for all closed subsets } F \textit{ of } \mathbb{C}.$$

Proof. Since $X_T(F) \subseteq E_T(F)$,

$$\theta X_T(F) \subseteq \theta E_T(F) = \theta(T-\lambda)E_T(F) = (S-\lambda)\theta E_T(F)$$

for every $\lambda \in \mathbb{C} \setminus F$. This shows that $\theta E_T(F) \subseteq E_S(F)$ and since $E_S(F) = H_S(F)$, by Proposition 11, the proof is complete. □

THEOREM 13. *Suppose that $T \in \mathcal{L}(X)$ is decomposable and that $S \in \mathcal{L}(H)$ is pure hyponormal. Then every linear operator $\theta : X \to H$ for which $\theta T = S\theta$ is necessarily continuous.*

Proof. Consider an arbitrary linear operator $\theta : X \to H$ satisfying $S\theta = \theta T$. To prove the continuity of θ, it suffices to construct a non-trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$. Indeed if we do so, since all factors $S - \lambda$ of $p(S)$ is injective by Proposition 10, we have $\mathfrak{S}(\theta) = \{0\}$.

From Proposition 12, we infer that $\theta X_T(F) \subseteq H_S(F)$ for all closed subsets F of $\mathbb{C}$. Since $X_T(F)$ is the spectral capacity and $H_S(F)$ is stable, by Lemma 7, there is a finite set Λ of $\mathbb{C}$ such that $\mathfrak{S}(\theta) \subseteq H_S(\Lambda)$. An application of the Stability Lemma to the sequence $T-\lambda$, where $\lambda \in \Lambda$, yields a polynomial p for which

$$\mathfrak{S}(\theta p(T)) = \mathfrak{S}(\theta p(T)(T-\lambda)) \quad \text{for every } \lambda \in \Lambda.$$

Since θ intertwines T and S, this means that by Lemma 6

$$((S-\lambda)p(S)\mathfrak{S}(\theta))^- = (p(S)\mathfrak{S}(\theta))^- \quad \text{for every } \lambda \in \Lambda.$$

Applying Mittag-Leffler Theorem, there exists a dense subspace $W \subseteq (p(S)\mathfrak{S}(\theta))^-$ for which $(S-\lambda)W = W$ for every $\lambda \in \Lambda$. This means that $W \subseteq E_S(\mathbb{C}\setminus\Lambda)$ by the definition of algebraic spectral subspaces. Since $W \subseteq \mathfrak{S}(\theta) \subseteq E_S(\Lambda)$, we obtain that

$$W \subseteq E_S(\Lambda) \cap E_S(\mathbb{C} \setminus \Lambda) = E_S(\emptyset).$$

Since S is pure hyponormal, S has no non trivial divisible subspace. Hence $E_S(\emptyset) = \{0\}$. Therefore we have, $W = \{0\}$. Consequently, $p(S)\mathfrak{S}(\theta) = \{0\}$. Hence θ is continuous. So the proof is complete. □

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