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# THE $(0,1)$–NORMAL SANDWICH PROBLEM

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Abstract. We study the question of whether a partial $(0,1)$-normal matrix has a non-symmetric normal completion. Matrix sandwich problems are an important and special case of matrix completion problems. In this paper, we give some properties for the $(0,1)$-normal matrices and some large classes that satisfies the normal sandwich completion.

## 1. Introduction

Let $M_1$ and $M_2$ be $m \times n$ $(0,1)$-matrices. We say that $M_1$ is dominated by $M_2$ ($M_2$ dominates $M_1$) denoted by $M_1 \leq M_2$ if for all $1 \leq i,j \leq n$, then

$$M_1(i,j) = 1 \quad \text{implies} \quad M_2(i,j) = 1.$$

Consider the following problem: For given $M_1 \leq M_2$ and a matrix property $\Phi$, is there an $m \times n$ $(0,1)$-matrix $M$ satisfying property $\Phi$ such that $M_1 \leq M \leq M_2$? This problem is called the $\Phi$-matrix sandwich problem and $M$ is called the $\Phi$-completion of $M_1$ in $M_2$. An alternate way of regarding a sandwich problem is to consider an $m \times n$ matrix $A$ with entries $(0,1,*)$ and to ask the question whether each $*$ entry (interpreted as "do not care") can be filled in by 0 or 1 such that the filled-in matrix $M$ satisfies property $\Phi$. The matrix $M$ is called a $\Phi$-completion of $A$.

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Graph sandwich problems are precisely those for which the matrices are the adjacency matrices of a graph. Hypergraph sandwich problems are precisely those for which the matrices are the (hyperedges-versus-vertices) incidence matrices of a hypergraph. Recently, sandwich problems that have been studied for graph properties include $NP$-complete sandwich results for interval graphs, chordal graphs, unit interval graphs, permutation and comparability graphs [6] and $k$-trees for general $k$, and polynomial sandwich algorithms for split graphs, cographs [6], unit interval graphs with bounded clique size, $k$-tree for fixed $k$ and graphs containing a homogeneous set.

Matrix sandwich problems are an important and special type of matrix completion problems. Matrix completion problems are generally defined over the real numbers that rather than simply $(0,1)$ and the unspecified entries are to be filled in so as to achieve a matrix with a desired numerical property. For example, the positive definite and semi-definite matrix completion problem [2], [3], [4], [6], the Euclidean distance matrix completion problem [1], band matrix completions [5], Jordan and Hessenbery matrix completions [7] have been studied. Throughout this paper every notation is the standard in general graph theory.

Obviously any symmetric matrix is normal, so that we can concentrate on non symmetric normal matrices. It easy to see that for $n \leq 2$ there is no non symmetric normal $(0,1)$-matrix of order $n$.

Since a matrix is normal if and only if it is permutation similar to a direct sum of irreducible normal matrices, we can focus our study on irreducible normal matrices.

A matrix $A$ of order $n \geq 2$ is said to be reducible if there exists a permutation matrix such that

$$PAP^T = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

where $B$ and $D$ are square matrices of order at least one. Otherwise $A$ is said to be irreducible. As usual $J$ denotes the all 1s matrix of the appropriate size. We say that the complement of a $(0,1)$-matrix $B$ is $J-B$ and we denote the complement of $B$ by $B^c$.

Let $R=(r_1,r_2,\cdots,r_n)$ and $C=(c_1,c_2,\cdots,c_n)$ be the row-sum and column-sum vectors of an $n\times n$ normal $(0,1)$-matrix $B$, where $r_i(c_i)$ denotes the $i$-th row(column, respectively) sum of $B$. It is easily seen that $R=C$. Recently, Wang and Zhang characterized the vectors $R$ that can arise as the row-sum vectors of $(0,1)$-normal matrices. Therefore, the following lemma is clear.

LEMMA 1.1. *The set of $n\times n$ non-symmetric irreducible patterns in the $(0,1)$-normal matrices is closed under*

*(1) permutation similarity,*

*(2) transposition,*

*(3) complementation.*

We say that two $(0,1)$-matrices $A$, $B$ are equivalent if $B$ can be obtained from $A$ via (finitely many) operations in Lemma 1.1. This yields an equivalence relation on the set of $n\times n$ non symmetric normal $(0,1)$-matrices. It is note that if $B$ is a reducible normal $(0,1)$-matrix, then $B^c$ is irreducible. Hence an equivalence class can contain both reducible and irreducible matrices. If all matrices in the equivalence class are irreducible, then the equivalence class is called an irreducible equivalence class. In general, it may be of interest to study $(0,1)$-matrices $B$ such that $B$ and $B^c$ are both irreducible. We denote the cell, $E_{ij}$, is a $(0,1)$-matrix that have 1 at the $(i,j)$-position and otherwise are all zeros.

The following Theorems are induced by simple calculations and the property of Lemma 1.1.

THEOREM 1.2. *Let $A$ be an non-symmetric irreducible $(0,1)$-normal* *matrix. If $a_{ii} = 0$ and the $i$-th row vector is same as the $i$-th column vector of $A$, then the matrix $A + E_{ii}$ is also non-symmetric irreducible normal.*

THEOREM 1.3. *Let $A$ be an non-symmetric irreducible $(0,1)$-normal matrix. If $a_{ij} = 0 = a_{ji}$ and the $\{i,j\}$-th row and column vectors are all same, then the matrix $A + E_{ij} + E_{ji}$ is also non-symmetric irreducible normal.*

THEOREM 1.4. *Let $B$ be an $(n-1) \times (n-1)$ non-symmetric irreducible $(0,1)$-normal matrix. If for each $\{i_1, i_2, \cdots, i_k\}$-th row and column vectors of $B$ and $1 \le k \le n-2$, the row sum vector $[r_{i_1}, r_{i_2}, \cdots, r_{i_k}]^T$ and the column sum vector $[c_{i_1}, c_{i_2}, \cdots, c_{i_k}]$ are same, then the matrix $A$ as followings:*

$$A = \left[\begin{array}{cccccccccc}
* & \vdots & 0 & \cdots & a_{1i_{(n-k+1)}} & a_{1i_{(n-k+2)}} & \cdots & a_{1i_{(n-1)}} & a_{1i_n} \\
\cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \vdots & & & & & & & \\
\vdots & \vdots & & & & & & & \\
0 & \vdots & & & & & & & \\
a_{i_{(n-k+1)}1} & \vdots & & & PBP^T & & & & \\
a_{i_{(n-k+2)}1} & \vdots & & & & & & & \\
\vdots & \vdots & & & & & & & \\
a_{i_{(n-1)}1} & \vdots & & & & & & & \\
a_{i_n 1} & \vdots & & & & & & &
\end{array}\right]$$

*where $P$ is a permutation matrix that reordered $\{i_1, i_2, \cdots, i_k\}$-th row and column vectors of $B$ to $\{i_{n-k+1}, i_{n-k+2}, \cdots, i_n\}$-th row and column vectors, $a_{1i_{(\cdot)}}$ and $a_{i_{(\cdot)}1}$ are all 1 and the $*$ is 0 or 1, is also non-symmetric irreducible normal.*

THEOREM 1.5. *Let $A$ be an non-symmetric irreducible $(0, 1)$-normal matrix. If $A$ has an $k \times k$-zero block $O_k$ as the $k \times k$ leading principle diagonal block (up to equivalence relation) and for the submatrices $C = [r_1, r_2, \cdots, r_k]^T$ and $D = [c_1, c_2, \cdots, c_k]$ of $B$, column sum vector of $C$ and row sum vector of $D$ is same, then the matrix $A + \{J_k \bigoplus O_{(n-k)}\}$ is also non-symmetric irreducible normal.*

## 2. The $(0, 1)$-normal matrix sandwich problem

Let $N_1$ and $N_2$ be $n \times n$ $(0, 1)$-normal matrices. We consider the following problem: For given $N_1 \leq N_2$, is there an $n \times n$ $(0, 1)$-normal matrix $N_*$ such that $N_1 \leq N_* \leq N_2$? This problem is called the normal matrix sandwich problem and $N_*$ is called a normal-completion of $N_1$ in $N_2$. Since we have already normal matrices as the lower and upper bounds in the dominate sequence $N_1 \leq N_* \leq N_2$, our main problem is the strong normal matrix sandwich completion such that $N_1 < N_* < N_2$.

The next example shows the form of our normal matrix sandwich problem.

EXAMPLE 2.1 Let

$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then $N_2$ dominates $N_1$ and there exists a strong normal completion $N_*$ of $N_1$ in $N_2$ where

$$N_* = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The following Theorem is very useful to characterize our main problem.

THEOREM 2.2. *Let $A$ be an $n \times n$ $(0,1)$-normal matrix. If $A$ is decomposed by $N_S$ and $N_C$ such that $A = N_S + N_C$ where $N_S$ is the symmetric part and $N_C$ is the non-symmetric part of $A$, then each entries $a_{ij}$ of $N_C$ is contained in the only one cycle of directed graph $G(N_C)$ of $N_C$.*

*Proof.* Suppose that there exists an nonzero entry $a_{ij}$ of $N_C$ that is not contained in any cycle of directed graph $G(N_C)$ of $N_C$. Since each row and column sum vector of $A$ and $N_S$ are same, $N_C$ must have the same row and column sum vectors. By the hypothesis, there exists an largest path $P_k$ in $G(N_C)$ that contains the edge $(v_i, v_j)$, i.e., not a cycle and that has the length $k$, that is $P_k = (v_1, v_2, \cdots, v_i, v_j, \cdots, v_k)$. By the simple calculation, we know that the adjacent matrix of $P_k$ has different row and column sum vectors. If there exists an edge $(v_k, v_1)$, it is contradiction to the fact that $P_k$ is a largest path and not a cycle. Thus, an nonzero entry $a_{ij}$ of $N_C$ must contained in any cycle of directed graph $G(N_C)$.

Now we assume that there exists an entry $a_{ij}$ of $N_C$ that is contained in the two cycle of $G(N_C)$. Then, by the above reason, it is contradiction. Therefore, $G(N_C)$ is the digraph that have only distinct simple cycles. The proof is completed. $\square$

EXAMPLE 2.3 Let $A$ be a $5 \times 5$ normal matrix such that

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then

$$A = N_S + N_C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In addition, we know that the adjacent matrix $N_C$ of $G(N_C)$ is decomposed by the two simple cycle's sum as following form:

$$N_C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The following lemma is found in [8].

LEMMA 2.4. *Let $A$ be symmetric. Then*

$$\begin{bmatrix} B & C \\ C^T & A \end{bmatrix} \in M_n(\mathbb{R})$$

*is normal if and only if $B$ is normal and $(B - B^T)C = 0$.*

THEOREM 2.5. *Let $A$ be an $n \times n$ $(0,1)$-matrix. If $A$ is represented as*

$$\begin{bmatrix} S_{kk} & \vdots & J_{k(n-k)} \\ J_{(n-k)k} & \vdots & N_{(n-k)(n-k)} \end{bmatrix}$$

*where $N_{(n-k)(n-k)}$ is any normal and $S_{kk}$ is any symmetric, then $A$ is normal.*

*Proof.* Suppose that $B$ is any $(n-k)\times(n-k)$ normal $(0,1)$-matrix. By Theorem 2.2, $B = N_S + N_C$. Thus,

$$\begin{aligned} B - B^T &= (N_S + N_C) - (N_S^T + N_C^T) \\ &= (N_S - N_S^T) + (N_C - N_C^T) \\ &= O + (N_C - N_C^T). \end{aligned}$$

We know that each row, column sum of the matrix $B - B^T$ is zero because $(N_C - N_C^T)$ is skew symmetric and $N_C$ has the property of Theorem 2.2. Therefore we have

$$(B - B^T) \cdot J_{k(n-k)} = O.$$

The proof is completed by Lemma 2.4 and the permutation similarity.
□

THEOREM 2.6. *Let $N_1$ and $N_2$ be $n \times n$ $(0,1)$-normal matrices as followings:*

$$N_1 = \begin{bmatrix} S_{kk}^L & \vdots & J_{k(n-k)} \\ J_{(n-k)k} & \vdots & N_{(n-k)(n-k)} \end{bmatrix}, \quad N_2 = \begin{bmatrix} S_{kk}^U & \vdots & J_{k(n-k)} \\ J_{(n-k)k} & \vdots & N_{(n-k)(n-k)} \end{bmatrix}$$

*where $S_{kk}^L$ and $S_{kk}^U$ are distinct symmetric $k \times k$ $(0,1)$-normal matrices and the others are all same in $N_1$ and $N_2$. If there is a strong completion $S_{kk}^*$ of the symmetric part $S_{kk}^L$ in $S_{kk}^U$, then*

$$N_* = \begin{bmatrix} S_{kk}^* & \vdots & J_{k(n-k)} \\ J_{(n-k)k} & \vdots & N_{(n-k)(n-k)} \end{bmatrix}$$

*is a strong normal completion of $N_1$ in $N_2$.*

*Proof.* Suppose that $S_{kk}^*$ is a strong normal completion of the symmetric part $S_{kk}^L$ in $S_{kk}^U$. Since $S_{kk}^L < S_{kk}^* < S_{kk}^U$, $N_*$ is a normal and satisfies the dominated sequence $N_1 < N_* < N_2$ by Theorem 2.5. Therefore, $N_*$ is a strong normal completion of $N_1$ in $N_2$. The proof is completed. □

EXAMPLE 2.7 Let $N_1$ and $N_2$ be $5 \times 5$ normal matrix as followings:

$$N_1 = \begin{bmatrix} 1 & 0 & \vdots & 1 & 1 & 1 \\ 0 & 0 & \vdots & 1 & 1 & 1 \\ \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ 1 & 1 & \vdots & 1 & 1 & 0 \\ 1 & 1 & \vdots & 0 & 1 & 1 \\ 1 & 1 & \vdots & 1 & 0 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 1 & \vdots & 1 & 1 & 1 \\ 1 & 1 & \vdots & 1 & 1 & 1 \\ \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ 1 & 1 & \vdots & 1 & 1 & 0 \\ 1 & 1 & \vdots & 0 & 1 & 1 \\ 1 & 1 & \vdots & 1 & 0 & 1 \end{bmatrix}.$$

Then there is a dominated sequence in the symmetric part $S_{22}$ such that

$$S_{22}^L = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad < \quad S_{22}^* = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad < \quad S_{22}^U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

By Theorem 2.6,

$$N_* = \begin{bmatrix} 1 & 1 & \vdots & 1 & 1 & 1 \\ 1 & 0 & \vdots & 1 & 1 & 1 \\ \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ 1 & 1 & \vdots & 1 & 1 & 0 \\ 1 & 1 & \vdots & 0 & 1 & 1 \\ 1 & 1 & \vdots & 1 & 0 & 1 \end{bmatrix}$$

is a strong normal completion of $N_1$ in $N_2$.

THEOREM 2.8. *Let $N_1$ and $N_2$ be $n \times n$ $(0,1)$-normal matrices, $N_1 < N_2$, and $K_* = N_2 - N_1$. If the directed graph $G(K_*)$ has a proper clique $c_K$ except loop which vertices are isolated to $G(N_C)$ or contained perfectly in the only one cycle of $G(N_C)$ of the cycle part $N_C$ in $N_1$, then there is a strong normal completion $N_1 + A(c_K)$ of $N_1$ in $N_2$ where $A(c_K)$ is the adjacent matrix of the proper clique $c_K$.*

*Proof.* Suppose that $K_*$ has a proper clique $c_K$ which vertices are contained in the two cycles $\gamma_1$ and $\gamma_2$ of $G(N_C)$ in $N_1$. If these are disjoint two cycles $\gamma_1$ and $\gamma_2$, then there exists a proper clique in $c_K$ which vertices are contained perfectly in the only one cycle. Hence we may assume that these two cycles $\gamma_1$ and $\gamma_2$ are connected. Then we have joined vertices $\{v_1, v_2, \cdots, v_i\}$ in the two cycles $\gamma_1$ and $\gamma_2$. Since joined vertices $\{v_1, v_2, \cdots, v_i\}$ do not consist a simple cycle, there exist some inner products of row $r_{(\cdot)}$ and column $c_{(\cdot)}$ in $N_1 + A(c_K)$ such that $r_p \cdot r_q \neq c_p \cdot c_q$ where the index $p$ is in the set of joined vertices and $q$ is in the others. Thus $N_1 + A(c_K)$ is not normal. By the simple calculation and Theorems in section 1, we know that $N_1 + A(c_K)$ is normal and satisfied the dominated sequence $N_1 < N_1 + A(c_K) < N_2$ when the vertices of proper clique $c_K$ are isolated to $G(N_C)$ or contained perfectly in the only one cycle of $G(N_C)$ of the cycle part $N_C$ in $N_1$. The proof is completed. $\square$

EXAMPLE 2.9 Let $N_1$ and $N_2$ be $5 \times 5$ normal matrix as followings:

$$N_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1_c \\ 0 & 0 & 0 & 1_c & 0 \\ 1 & 1_c & 0 & 1 & 1 \\ 1 & 0 & 1_c & 1 & 1 \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1_k & 1_k & 0 & 1_c \\ 1 & 1_k & 1_k & 1_c & 0 \\ 1 & 1_c & 0 & 1 & 1 \\ 1 & 0 & 1_c & 1 & 1 \end{bmatrix}.$$

Then

$$K_* = N_2 - N_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1_k & 1_k & 0 & 0 \\ 1 & 1_k & 1_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $G(K_*)$ has a proper clique $c_K$. Therefore,

$$
\begin{aligned}
N_* &= N_1 + A(c_K) \\
&= \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1_c \\ 0 & 0 & 0 & 1_c & 0 \\ 1 & 1_c & 0 & 1 & 1 \\ 1 & 0 & 1_c & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1_k & 1_k & 0 & 0 \\ 0 & 1_k & 1_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1_k & 1_k & 0 & 1_c \\ 0 & 1_k & 1_k & 1_c & 0 \\ 1 & 1_c & 0 & 1 & 1 \\ 1 & 0 & 1_c & 1 & 1 \end{bmatrix}
\end{aligned}
$$

is a strong normal completion of $N_1$ in $N_2$.

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