JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **16**, No.1, June 2003

AN ESCAPE CRITERION FOR THE COMPLEX POLYNOMIAL, WITH APPLICATIONS TO THE DEGREE-n BIFURCATION SET

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Abstract. Let $P_c(z) = z^n + c$ be a complex polynomial with an integer $n \geq 2$. We derive a criterion that the critical orbit of P_c escapes to infinity and investigate its applications to the degree-n bifurcation set. The intersection of the degree-n bifurcation set with the real line as well as with a typical symmetric axis is explicitly written as a function of n. A well-defined escape-time algorithm is also included for the improved construction of the degree-n bifurcation set.

1. Introduction

An escape criterion has been investigated by Devaney([4−7]) and other researchers for the complex quadratic polynomial $z^2 + c$. In this paper, we extend the investigation to a more general complex polynomial $P_c(z) = z^n + c$ with $n \geq 2$. The intersection of the degree-n bifurcation set([4, 8]) with the real line is introduced by Carleson and Gamelin([3]) for the case of $n = 2$. Using the escape criterion and the symmetry([8]) of the degree-n bifurcation set, the intersection with the real line as well as with a typical symmetric axis will be pursued for more general cases with $n \geq 2$. An escape-time algorithm([2]) constructing the degree-n bifurcation set is also established on the basis of the escape criterion presented here. Its implementation is shown in

*Supported by the research fund of Dankook University in 2003.
Received by the editors on March 17, 2003.
2000 *Mathematics Subject Classifications* : Primary 00A05, 00A69, 37F45.
Key words and phrases: bifurcation, degree-n bifurcation set, Mandelbrot set, intersection, escape criterion.

Figure 1 for typical degree-n bifurcation sets. The following notations and symbols are used throughout the paper.

$\boldsymbol{C}$: set of all complex numbers.

$\boldsymbol{R}$: set of all real numbers.

$\boldsymbol{N}$: set of all natural numbers.

$f^k(z) = f \circ f^{k-1}(z)$: k-fold composite map of f at z with $f^0(z) = z$.

Definition 1.1. Let $P_c(z) = z^n + c$ for an integer $n \geq 2$, with $c,\ z \in \boldsymbol{C}$. Then the *degree-n bifurcation set* is defined to be the set

$$\boldsymbol{M} = \left\{ c \in \boldsymbol{C} : \lim_{k\to\infty} P_c^k(0) \neq \infty \right\}.$$

If $n = 2$, it reduces to the Mandelbrot set([2–8, 10, 11]).

2. An escape criterion for the complex polynomial

It is not convenient to construct $\boldsymbol{M}$ using Definition 1.1 since the critical orbit may contain the infinite number of terms. The following theorem states the well-defined limit behavior for the boundedness of the critical orbit of the complex polynomial $z^n + c$.

Theorem 2.1. *Let $P_c(z) = z^n + c$ for $n \in \boldsymbol{N} - \{1\}$, with $c,\ z \in \boldsymbol{C}$. Then*

$$\lim_{k\to\infty} P_c^k(0) \neq \infty \ if \ and \ ony \ if \ |{P_c}^k(0)| \leq 2^{\frac{1}{n-1}} \ for \ all \ k \geq 1$$

Proof. If $|c| = |P_c(0)| > 2^{\frac{1}{n-1}}$, one can show by induction on $k \geq 1$ that

$$|{P_c}^{k+1}(0)| \geq |c|(|c|^{n-1} - 1)^{n^{k-1}} \tag{1.1}$$

According to Eqn.(1.1), we have $|{P_c}^k(0)| \to \infty$ as $k \to \infty$. It suffices to show that the converse is true. Now suppose that there exists a positive integer $m \geq 1$ such that $|{P_c}^m(0)| = 2^{\frac{1}{n-1}} + \delta > 2^{\frac{1}{n-1}}$ with $\delta > 0$. If $|c| = |P_c(0)| > 2^{\frac{1}{n-1}}$, then we obtain $|{P_c}^k(0)| \to \infty$ as $k \to \infty$. If $|c| = |P_c(0)| \leq 2^{\frac{1}{n-1}}$, then

$$|P_c^{m+1}(0)| \geq |P_c^{m}(0)|^n - |c| \geq (2^{\frac{1}{n-1}} + \delta)^n - 2^{\frac{1}{n-1}}$$
$$= 2^{\frac{n}{n-1}}(1 + \delta\, 2^{\frac{-1}{n-1}})^n - 2^{\frac{1}{n-1}} \geq 2^{\frac{1}{n-1}} + 2n\delta$$

Proceeding by induction, we obtain $|P_c^{m+k}(0)| \geq 2^{\frac{1}{n-1}} + (2n)^k\delta \to \infty$ as $k \to \infty$, completing the proof. □

The value $2^{1/(n-1)}$ generalizes the escape criterion for the complex polynomial $z^n + c$ and it certainly reduces to the value 2 for the well-known Mandelbrot set. Theorem 2.2 follows immediately from the result of Theorem 2.1.

THEOREM 2.2. *Let $n \geq 2$ be an integer. Then we have the degree-n bifurcation set*

$$\boldsymbol{M} = \{c \in \boldsymbol{C} : \ |P_c^{k}(0)| \leq 2^{\frac{1}{n-1}} \ \textit{for all}\ k \geq 1\} \subset \{c \in \boldsymbol{C} : |c| \leq 2^{1/(n-1)}\}.$$

3. Applications and concluding remarks

From the result of Section 2, some properties of the degree-n bifurcation set are deduced including the well-defined escape-time algorithm constructing the degree-n bifurcation set. In Theorem 3.1, the intersection of the degree-n bifurcation set with the real line is explicitly written as a function of n using the results of Theorem 2.2. Theorem 3.2 gives an extended result of Theorem 3.2 by rotating the real line by an appropriate angle $\theta_k = 2k\pi/(n-1)$ through the origin in the complex plane ([1]).

THEOREM 3.1. *Let $n \geq 2$ be a given integer and $\rho = (1-1/n)(1/n)^{1/(n-1)}$. Then*

$$\boldsymbol{M} \cap \boldsymbol{R} = \begin{cases} [-2^{\frac{1}{n-1}},\ \rho] & \textit{if } n \textit{ is even} \\ [-\rho,\ \rho] & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Consider $c \in [-2^{\frac{1}{n-1}},\ 2^{\frac{1}{n-1}}]$ and let x be a real fixed point of P_c such that

$$P_c(x) = x^n + c = x.$$

When n is even, it can be shown that $c = x - x^n$ assumes its maximum ρ at $x = (1/n)^{1/(n-1)}$. Hence it suffices to consider $c \in [-2^{\frac{1}{n-1}}, \rho]$. Let $a > 0$ be the largest real fixed point of P_c such that $a^n + c = a$. Indeed, one can show that $a = 2^{\frac{1}{n-1}}$. For $0 < c = P_c(0) = a - a^n \leq \rho < a$, it follows that

$$0 < {P_c}^k(0) < a^n + c = a$$

by induction on $k \geq 1$. Hence such $c \in \boldsymbol{M}$. For $-2^{\frac{1}{n-1}} = -a \leq c = P_c(0) \leq 0$, it is clear that for all $k \geq 1$ with even n

$${P_c}^{k+1}(0) = {P_c}^k(0)^n + c \geq c \geq -a,$$

$$-a \leq c \leq {P_c}^2(0) = P_c(0)^n + c = |P_c(0)|^n + c \leq a^n + c = a.$$

Proceeding by induction, $0 \leq |{P_c}^k(0)| \leq a$ for all $k \in \boldsymbol{N}$ with $-2^{\frac{1}{n-1}} \leq c \leq 0$. Hence such $c \in \boldsymbol{M}$. As a result, $\boldsymbol{M} \cap \boldsymbol{R} = [-2^{\frac{1}{n-1}}, \rho]$.

When n is odd, due to symmetry studied by Geum and Kim([8]), it suffices to consider for $c > 0$. For $0 < \rho < c \leq 2^{\frac{1}{n-1}}$, we have $c = P_c(0) > \rho > 0$. Proceeding by induction on $k \geq 2$, we have ${P_c}^{k+1}(0) > {P_c}^k(0) > c > \rho$. Thus $\{P_c^k(0)\}$ is monotone increasing and not bounded above, from which $\lim_{k->\infty} {P_c}^k(0) = \infty$. Hence such $c \notin \boldsymbol{M}$. For $0 \leq c \leq \rho$, let $a > 0$ be the largest real fixed point of P_c such that $a^n + c = a$. Proceeding by induction on $k \geq 1$, we have $0 \leq {P_c}^k(0) < a$ for all $k \in \boldsymbol{N}$. Hence such $c \in \boldsymbol{M}$. Consequently, the symmetry shows that $\boldsymbol{M} \cap \boldsymbol{R} = [-\rho, \ \rho]$. □

Remark 1. If $n = 2$, then Theorem 2.2 gives the result of Carleson and Gamelin ([3]).

The symmetry of the degree-n bifurcation set, together with the result of Theorem 3.1, leads immediately to the following Theorem 3.2 which describes the intersection of the degree-n bifurcation set with a typical symmetric axis as a function of n.

THEOREM 3.2. *Let* $n \in \boldsymbol{N} - \{1\}$ *be given and* $\rho = (1-1/n)(1/n)^{1/(n-1)}$. *For each* $k \in \{0, 1, 2, \cdots, n-2\}$, *let* $\theta_k = 2k\pi/(n-1)$ *and* $\boldsymbol{\Omega}_k = \{c \in \boldsymbol{C} : c = x + iy,\ x \in \boldsymbol{R},\ y = x \tan\theta_k,\ with\ i = \sqrt{-1}\}$ *denote a symmetric axis of* $\boldsymbol{M}$. *Then we obtain*

$$\boldsymbol{M} \cap \boldsymbol{\Omega}_k = \{c \in \boldsymbol{C} : c = x + iy,\ a\cos\theta_k \le x \le \rho\cos\theta_k\ y = x\tan\theta_k\},$$

where $a = \begin{cases} -2^{\frac{1}{n-1}}\ if\ n\ is\ even \\ -\rho\ if\ n\ is\ odd. \end{cases}$

As a result of the escape criterion described in Theorem 2.2, we establish the following improved escape-time algorithm ([2]) which easily handles the infinite number of terms in calculating the critical orbit.

ALGORITHM 1. *Let* $P_c(z) = z^n + c$ *for an integer* $n \ge 2$, *with* $c,\ z \in \boldsymbol{C}$. *Let* BGCO *denote the color number of escaping points and* n_c *denote the maximum number of indexed color numbers. Then the construction algorithm of the degree-*n *bifurcation set is described below:*

Step 1. *Choose a maximum number of iterations,* ITER *and confine a region contained in* $\{c \in \boldsymbol{C} : |c| \le 2^{1/(n-1)}\}$.

Step 2. *For each point* c *in the confined region, compute the first* ITER *points in the critical orbit of* P_c *and store the last* $n_c - 1$ *points among them.*

Step 3. *If* $|{P_c}^i(0)| > 2^{1/(n-1)}$ *for some* $i \le$ ITER, *then stop the iteration and paint the grid point* c *in a color of* BGCO.

Step 4. *If* $|{P_c}^i(0)| \le 2^{1/(n-1)}$*for all* $i \le$ ITER, *then*

1) *compute the period* k *of the orbit from the stored points.*

2) (a) *if* $1 \le k \le n_c - 1$, *then paint the grid point* c *in a color of index number* k

(b) *else paint the grid point* c *in a color of* BGCO.

Definition 3.1. The *attracting period-k component* ([9]) is defined as the set

$$\boldsymbol{M}_k{}' = \{c \in \boldsymbol{C} : there\, exists\, z_0\, such\, that\, P_c{}^k(z_0) = z_0, \left| \frac{d}{dz} P_c{}^k(z) \right|_{z=z_0} < 1\}.$$

On the basis of Algorithm 1, typical degree-n bifurcation sets are constructed and shown in Figure 1 in the c-parameter plane for $2 \leq n \leq 7$. The component $\boldsymbol{M}_{\boldsymbol{k}}'$ is identified by a number k and shaded in different patterns or colors. It can be easily shown that the interval $[-2^{1/(n-1)},\ \rho]$ or $[-\rho,\ \rho]$ approaches $[-1, 1]$ as n tends to infinity. Although details of our elaborate numerical experiments are not shown here, careful measurements from Figure 1 show a good agreement with the result of Theorem 3.1. Although the value $2^{1/(n-1)}$ nicely characterizes the escape criterion, we require high number of iterations as well as sufficient precision digits to check the criterion near the boundary of $\boldsymbol{M}$.

A future study is to estimate the area of the degree-n bifurcation set by counting the pixels whose critical orbits are judged to be bounded on the basis of the escape criterion investigated here, assuming that each pixel represents a square region of the complex plane.

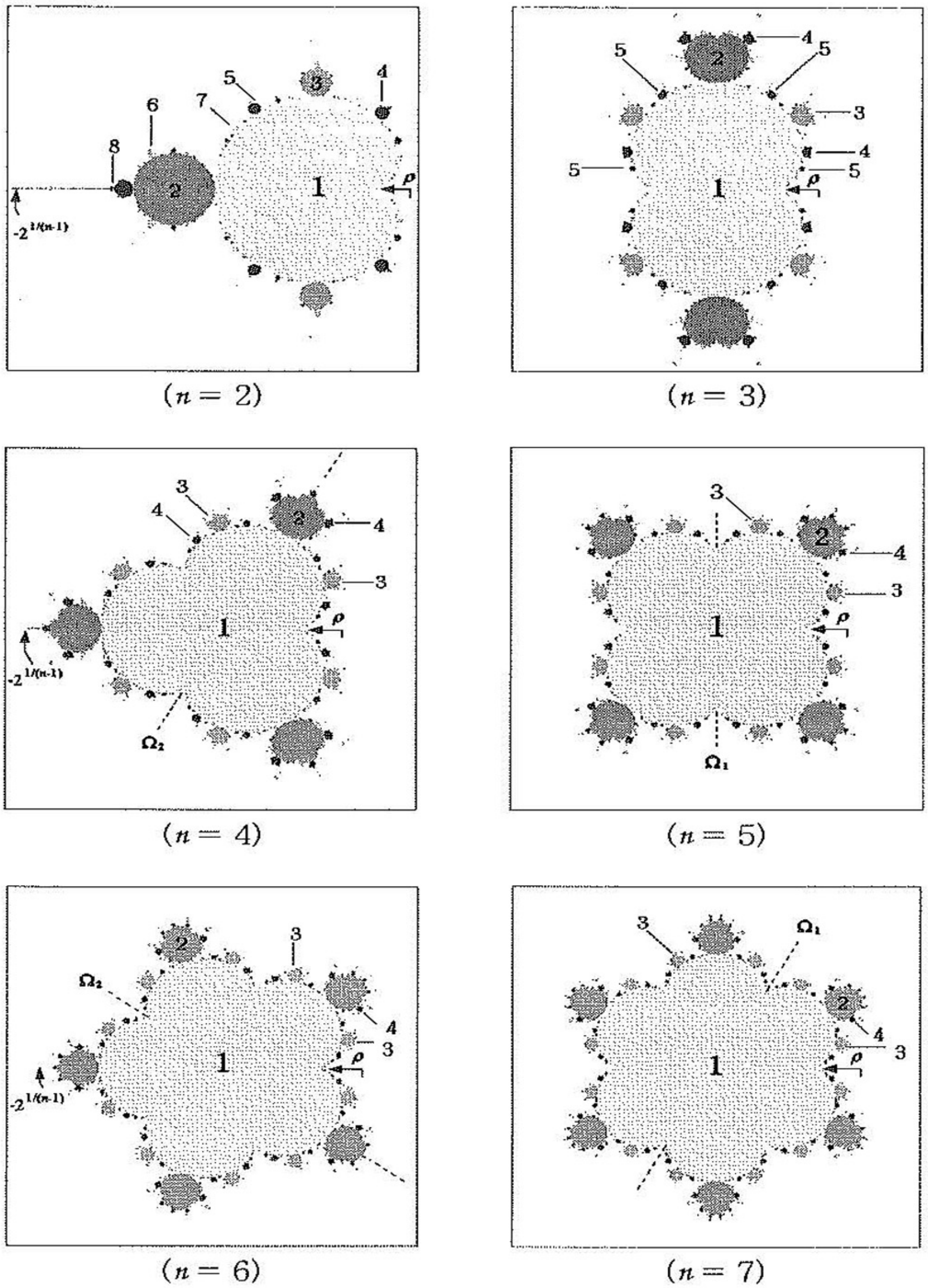

FIGURE 1. Typical degree-n bifurcation sets

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