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ON LIFT OF HOMOTOPIC MAPS

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Abstract. By considering a hyperspace $CL(X)$ of a Hausdorff space X with the Vietoris topology [6] also called the finite topology and treating X as a subspace of $CL(X)$ with the natural embedding, it is obtained that homotopic maps $f,\ g : X \to Y$ are lifted to homotopic maps on the respective hyperspaces.

1. Introduction

Throughout the paper, spaces are Hausdorff and maps are continuous. Symbols $X, Y, Z, \cdots$, are used for spaces and $f, g, \cdots$ are used for maps between the spaces.

For a space X, $CL(X)$ denotes the collection of all non-empty closed subsets of X. The set $CL(X)$ equipped with some topology is called a *hyperspace* of X. Various topologies are given to $CL(X)$ and the study of hyperspaces is one of the active area of researches in topology nowadays. Various workers including E. Michael, H. Attouch, U. Mosco, R. Wets, J. Fell, and G. Beer have contributed in this area.

Among various topologies defined on $CL(X)$, the *Vietoris* topology also called the *finite* topology or the *exponential* topology is one of the most well studied topologies on $CL(X)$. Fundamentals on the Vietoris topologies can be found in [1, 6] and in the recent monograph of Klein and Thompson [5].

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A subbase for the Vietoris topology is given by the family $\{V^+ \mid V \text{ is open in } X\}$ and $\{V^- \mid V \text{ is open in } X\}$. The sets V^+ and V^- are described as below:

$$V^+ = \{F \in CL(X) \mid F \subset V\},$$
$$V^- = \{F \in CL(X) \mid F \cap V \neq \phi\}.$$

We denote the Vietoris topology by $\mathcal{T}_V$ in the sequel.

For a metrizable space X, an element $x \in X$ gives rise to a map $d_x : CL(X) \to \mathbb{R}$ defined by taking F to $d(x, F)$ for $F \in CL(X)$.

For a metric space X, the Vietoris topology is the smallest topology on $CL(X)$ which makes each d_x continuous for all admissible metric d on X ([2]).

In this paper we consider $CL(X)$ with the Vietoris topology $\mathcal{T}_V$. By embedding X into $CL(X)$ in the natural way, we obtain in Section 2 that for T_4 spaces X and Y a continuous bijective map $f : X \to Y$ has a continuous lift

$$Cf : CL(X) \to CL(Y)$$

over hyperspaces.

We further obtain that if f is homotopic to g then the lifted maps Cf and Cg are also homotopic.

Recall that $f, g : X \to Y$ are *homotopic* if there exists a continuous map $H : X \times [0,1] \to Y$ satisfying

$$H(x, 0) = f(x),$$
$$H(x, 1) = g(x)$$

for all $x \in X$. A homotopy $H : X \times [0,1] \to Y$ can be regarded as a continuous one parameter family of maps $h_t : X \to Y$ for $t \in [0,1]$,

$$h_t(x) = H(x, t).$$

For terms and notations not explained here we refer to [1, 3, 7].

2. Lift of homotopic maps on hyperspaces

Throughout the section, spaces are Hausdorff. The space X can be treated as a subspace of $CL(X)$ by natural embedding $e : X \to CL(X)$ defined by $e(x) = \{x\}$.

The following proposition shows that a certain map $f : X \to Y$ between T_4 spaces can be lifted to the respective hyperspaces continuously in an obvious manner.

PROPOSITION 2.1. *Let X and Y be T_4 spaces and $f : X \to Y$ a continuous bijective map. Then the map $Cf : (CL(X), \mathcal{T}_V) \to (CL(Y), \mathcal{T}_V)$, defined by*

$$Cf(A) = \overline{f(A)}, \qquad A \in CL(X),$$

is continuous.

Proof. To show that the map $Cf : CL(X) \to CL(Y)$ is continuous, it is enough to prove that the inverse images of subbasic open sets of $CL(Y)$ are open in $CL(X)$. Subbasic open sets of $CL(Y)$ are given by E^+ and E^-, where

$$E^+ = \{F \in CL(Y) \mid F \subset E\},$$
$$E^- = \{F \in CL(Y) \mid F \cap E \neq \phi\}.$$

To show that $(Cf)^{-1}(E^+)$ is open in $CL(X)$, take $A \in (Cf)^{-1}(E^+)$. Then $\overline{f(A)} \subset E$. Normality of Y provides an open set U satisfying $\overline{f(A)} \subset U \subset \overline{U} \subset E$. Take $V = f^{-1}(U)$. Then V is an open set of X. It is seen that $A \in V^+ \subset (Cf)^{-1}(E^+)$.

To show that $(Cf)^{-1}(E^-)$ is open in $CL(X)$, take $A \in (Cf)^{-1}(E^-)$. Then $\overline{f(A)} \cap E \neq \phi$. Take a point $p \in \overline{f(A)} \cap E$. Since X is regular, there is an open set U satisfying

$$p \in U \subset \overline{U} \subset E.$$

Then we have

$$A \cap f^{-1}(U) \neq \phi.$$

Take $V = f^{-1}(U)$. Then it is enough to show that

$$A \in V^- \subset (Cf)^{-1}(E^-).$$

It is clear that $A \in V^-$. To show the other containment, take $B \in V^-$. Then we get $B \cap f^{-1}(U) \neq \phi$. This implies that $B \in (Cf)^{-1}(E^-)$, and completes the proof. □

In the following theorem, we obtain that if f is homotopic to g then the lifted maps Cf and Cg are also homotopic.

THEOREM 2.2. *Let X and Y be T_4 spaces and $f, g : X \to Y$ continuous bijective maps with $X \times [0, 1]$ a normal space. If f is homotopic to g, then Cf, $Cg : (CL(X), \mathcal{T}_V) \to (CL(Y), \mathcal{T}_V)$ are also homotopic.*

Proof. Let $H : X \times [0, 1] \to Y$ be a homotopy from f to g. Define a map $\mathcal{H} : CL(X) \times [0, 1] \to CL(Y)$ by

$$\mathcal{H}(A, t) = \overline{h_t(A)},$$

where $h_t : X \to Y$ is defined by

$$h_t(x) = H(x, t)$$

for all $x \in X$ and all $t \in [0, 1]$. It is clear that

$$H(A, 0) = Cf(A),$$
$$H(A, 1) = Cg(A)$$

for all $A \in CL(X)$.

To show that $\mathcal{H}$ is a homotopy between Cf and Cg, it remains to show that $\mathcal{H} : CL(X) \times [0,1] \to CL(Y)$ is continuous. To prove that $\mathcal{H}$ is continuous, we show that the inverse images of subbasic open sets of $CL(Y)$ are open in $CL(X) \times [0,1]$. Subbasic open sets of $CL(Y)$ are given by E^+ and E^-, where

$$\begin{aligned} E^+ &= \{F \in CL(Y) \mid F \subset E\}, \\ E^- &= \{F \in CL(Y) \mid F \cap E \neq \phi\}. \end{aligned}$$

Take $(A,t) \in \mathcal{H}^{-1}(E^+)$. Then $\overline{h_t(A)} \subset E$ and so $\overline{H(A \times \{t\})} \subset E$. Normality of Y provides an open set U satisfying

$$\begin{aligned} \overline{H(A \times \{t\})} &\subset U \subset \overline{U} \subset E, \\ H(A \times \{t\}) &\subset \overline{H(A \times \{t\})} \subset U, \\ A \times \{t\} &\subset H^{-1}(U). \end{aligned}$$

Normality of $X \times [0,1]$ gives open sets V and W of X and $[0,1]$, respectively, satisfying

$$A \times \{t\} \subset V \times W \subset \overline{V} \times \overline{W} \subset H^{-1}(U).$$

It is seen that $(A,t) \in V^+ \times W \subset \mathcal{H}^{-1}(E^+)$ implies that $\mathcal{H}^{-1}(E^+)$ is open.

Next take $(A,t) \in \mathcal{H}^{-1}(E^-)$. Then $\overline{h_t(A)} \cap E \neq \phi$. Take $y \in h_t(A) \cap E$. Since E is open, it is a neighborhood of $y \in Y$. Hence $h_t(A) \cap E \neq \phi$ or $H(A \times \{t\}) \cap E \neq \phi$. Let $x \in X$ be an element such that $H(x,t) \in H(A \times \{t\}) \cap E$, $H(x,t) \in E$ or $(x,t) \in H^{-1}(E)$. Since $X \times [0,1]$ is regular, there are open sets U and V satisfying

$$(x,t) \in U \times V \subset \overline{U} \times \overline{V} \subset H^{-1}(E).$$

We show that

$$(A,t) \in U^- \times V \subset \mathcal{H}^{-1}(E^-).$$

It is clear that $(A, t) \in U^- \times V$.

To show the other containment, take $(B, s) \in U^- \times V$. Then $B \cap U \neq \phi$ and $s \in V$. Take $p \in B \cap U$. Then $H(p, s) \in H(U \times V) \subset E$. So $H(B \times \{s\}) \cap E \neq \phi$ implies $\overline{H(B \times \{s\})} \cap E \neq \phi$ or $\overline{h_s(B)} \in E^-$, which implies $(B, s) \in \mathcal{H}^{-1}(E^-)$. This completes the proof. □

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