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ON THE FI–EXTENDING MODULES

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Abstract. In this paper, we study properties of a free normalizing extension ring of a FI-extending ring. We develop properties of formal triangular matrix rings and FI-extending rings.
Several results on the quasi-extending modules are obtained.

1. Introduction and FI-extending modules

All the rings considered will be associative rings with identity and all the modules considered will be unital modules. In recent years the theory of extending modules and rings has come to play an important role in the theory of rings and modules. Recall that a module M is called an extending (also known as a CS) module if every submodule of M is essential in a direct summand. Although this generalization of injectivity is very useful, it does not satisfy some important properties. For example, direct sum of extending modules are not necessarily extending, and full or upper triangular matrix rings over right extending rings are not necessarily right extending. Much work has been done on finding necessary and sufficient conditions to ensure that the extending property is preserved under various extensions (cf. [3]).

We say that a module is FI-extending if every fully invariant submodule is essential in a direct summand. The class of fully invariant

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submodules includes many of the significant submodules of a module (e.g., the Jacobson radical, the socle, the singular submodule, etc.).

Observe that when R is considered as a right R-module, the two sided ideals of R are exactly the fully invariant submodules of R_R. Hence we say R is a right (left) FI-extending ring if every ideal is right (left) essential in an idempotent generated right (left) ideal of R. Observe that every prime ring is right and left FI-extending.

Recall that a submodule X of a module M is called fully invariant if for every $h \in \mathrm{End}_R(M), h(X) \subseteq X$. If M is an R-module and $A \subseteq M$, then we say $A \leq M, A \leq^e M \quad A \lhd M$, and $E(M)$ to denote that A is a submodule, an essential submodule, a fully invariant submodule, and the injective hull of M, respectively.

If $M = R$, then $A \leq_r R$ ($A \leq_l R$) and $A \leq_r^e R$ ($A \leq_l^e R$) denote that A is a right (left) ideal of R and that A is right (left) essential in R.

If $\emptyset \neq X \subseteq R$, then $l(X)$ and $r(X)$ denote the left and right annihilators of X in R, respectively. Let $e = e^2 \in R$. Then e is called a left(right) semicentral idempotent if $xe = exe (ex = exe)$ for all $x \in R$ [2]. The set of all left (right) semicentral idempotents is denoted by $S_l(R)$ ($S_r(R)$).

Let S and R be two given rings and M a left S, right R bimodule. The formal triangular matrix ring $T = \begin{bmatrix} S & M \\ 0 & R \end{bmatrix}$ has as its elements formal matrices $\begin{bmatrix} a & m \\ 0 & r \end{bmatrix}$ where $a \in S, r \in R$ and $m \in M$ with addition given by coordinatewise and multiplication given by

$$\begin{bmatrix} a & m \\ 0 & r \end{bmatrix} \begin{bmatrix} a' & m' \\ 0 & r' \end{bmatrix} = \begin{bmatrix} aa' & am' + mr' \\ 0 & rr' \end{bmatrix} \qquad ([6]).$$

When R is a subring of the ring S, and R and S have the same identity 1, the ring S is a free normalizing extension of R if there is a finite set

$\{a_1, a_2, \cdots, a_n\} \subset S$ such that $a_1 = 1, S = Ra_1 + \cdots + Ra_n,\ a_iR = Ra_i$ for all $i = 1, 2, \cdots, n$ and S is free with basis $\{a_1, a_2, \cdots, a_n\}$ as both a right and left R-module [4].

PROPOSITION 1.1. *Let R be a ring and S be a free normalizing extension ring of R. Suppose that X is a right ideal of R and $e = e^2 \in R$ such that $X \leq^e eR$. Then $XS \leq^e eRS$.*

Proof. Let $S = a_1R + a_2R + \cdots + a_nR$, where $\{a_1, a_2, \cdots, a_n\}$ is a free basis of S over R and $a_1 = 1$. Let $e(r_1a_1 + r_2a_2 + \cdots + r_na_n) \neq 0$. If $er_1 \neq 0$, then $0 \neq er_1t_1 \in X$ for some $t_1 \in R$. Let $r_2a_2t_1 = r_2\overline{t_1}a_2 \neq 0$ and $er_2\overline{t_1} \neq 0$. Then $0 \neq er_1\overline{t_1}\ \overline{t_2} \in X$ for some $\overline{t_2} \in R$. Let $\overline{t}a_2 = at_2$. Inductively we see that $0 \neq e(r_1a_1 + r_2a_2 + \cdots + r_na_n)t_1t_2\cdots t_n \in XS$ for some $t_1, t_2, \cdots, t_n \in R$. Thus $XS \leq^e eRS$. □

THEOREM 1.2. *Let R be a right FI-extending ring and S a free normalizing extension ring of R. Every ideal of S has the form IS for some ideal I of R. Then S is a right FI-extending ring.*

Proof. Let I be an ideal of S. There exists an ideal X of R such that $I = XS$. By Proposition 1.1, there exists $e = e^2 \in R$ such that $XS \leq^e eRS$. Since $eRS = eS$ is a direct summand of S_s, S is a right FI-extending ring. □

The following corollaries follow immediately from the above theorem.

COROLLARY 1.3. ([2]) *Let $M_n(R)$ be the full ring of $n \times n$ matrices over the ring R. If R is a right FI-extending ring, then $M_n(R)$ is right FI-extending for all positive integer n.*

Proof. $M_n(R)$ is a free normalizing extension of R and satisfies the assumption of Theorem 2.1. □

Corollary 1.4. *Let R be a right FI-extending ring. Then crossed products [5] $R*G$ where G is a finite group and $|G|^{-1}\in R$ is a right FI-extending ring.*

Proof. $R*G$ is a free normalizing extension of R and satisfies the assumption of Theorem 2.1. □

Theorem 1.5. *Let M be an $S-R$-bimodule and $\begin{bmatrix} S & M \\ 0 & R \end{bmatrix}$ a formal triangular matrix ring and M be a faithful S-module. Then T is a right FI-extending ring if and only if R_R is right FI-extending and for each $S-R$ submodule K of M, there exists $e=e^2\in S$ such that $K\leq^e eM$.*

Proof. ($\Rightarrow$) Assume that T is a right FI-extending ring.

$R\cong \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\in S_r(T)$. By [1], R is a right FI-extending ring. Now let $0\neq K$ be an $S-R$ submodule of M. Then $\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}\lhd T$, i.e., $\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ is an ideal of T.

For $\begin{bmatrix} S & M \\ 0 & R \end{bmatrix}\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & SK \\ 0 & 0 \end{bmatrix}\subseteq\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ and

$\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S & M \\ 0 & R \end{bmatrix}=\begin{bmatrix} 0 & KR \\ 0 & 0 \end{bmatrix}\subseteq\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$.

Since T is right FI-extending, there exists $a=a^2\in T$ such that $\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}\leq^e aT$. Let $a=\begin{bmatrix} e & n \\ 0 & c \end{bmatrix}$ where $e\in S, n\in M$ and $c\in R$.

$a^2=\begin{bmatrix} e & n \\ 0 & c \end{bmatrix}\begin{bmatrix} e & n \\ 0 & c \end{bmatrix}=\begin{bmatrix} e^2 & en+nc \\ 0 & c^2 \end{bmatrix}=\begin{bmatrix} e & n \\ 0 & c \end{bmatrix}$. $e^2=e$ and $c^2=c$.

Suppose $c\neq 0$. $\begin{bmatrix} e & n \\ 0 & c \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}=\begin{bmatrix} 0 & nc \\ 0 & c \end{bmatrix}\neq 0$, $\begin{bmatrix} 0 & nc \\ 0 & c \end{bmatrix}\in aT$. There exists $\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\in T$ such that $0\neq\begin{bmatrix} 0 & nc \\ 0 & c \end{bmatrix}\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}=\begin{bmatrix} 0 & ncb \\ 0 & cb \end{bmatrix}\in\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$.

This implies that $ncb\neq 0$ and $cb=0$, a contradiction. Hence $c=0$.

Thus there exists $e = e^2 \in S$ and $n \in M$ such that $en = n$. $\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} \leq^e \begin{bmatrix} e & n \\ 0 & 0 \end{bmatrix} T$.

Let $0 \neq x \in eM$. $0 \neq \begin{bmatrix} e & n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} e & n \\ 0 & 0 \end{bmatrix} T$. There exists $\begin{bmatrix} a & m \\ 0 & r \end{bmatrix} \in T$ such that

$$0 \neq \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & m \\ 0 & r \end{bmatrix} = \begin{bmatrix} 0 & xr \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} .$$

This implies that $0 \neq xr \in K$. Therefore $K \leq^e eM$.

($\Leftarrow$) Assume R_R is FI-extending and for each $S-R$ submodule K of M, there exists $e = e^2 \in S$ such that $K \leq^e eM$. Let $0 \neq I \lhd T$. Then there exists $X \lhd S, Y \lhd R$ and K a $S-R$ submodule of M such that $MY \subseteq K, XM \subseteq K$, and $I = \begin{bmatrix} X & K \\ 0 & Y \end{bmatrix}$. Since K is an $S-R$ submodule of M, there exists $e = e^2 \in S$ such that $K \leq^e eM$. Since R is right FI-extending, there exists $c = c^2 \in R$ such that $Y \leq^e cR$. We will prove that $I \leq^e \begin{bmatrix} e & 0 \\ 0 & c \end{bmatrix} T$. Let $0 \neq \begin{bmatrix} ea & em \\ 0 & cr \end{bmatrix} \in \begin{bmatrix} e & 0 \\ 0 & c \end{bmatrix} T$ where $a \in S, r \in R$ and $m \in M$. Assume $em \neq 0$. There exists $b \in R$ such that $0 \neq emb \in K$. If $crb = 0$, then $0 \neq \begin{bmatrix} ea & em \\ 0 & cr \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & emb \\ 0 & 0 \end{bmatrix} \in I$.

If $crb \neq 0$, then there exists $d \in R$ such that $0 \neq crbd \in Y$. Hence $0 \neq \begin{bmatrix} ea & em \\ 0 & cr \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & bd \end{bmatrix} = \begin{bmatrix} 0 & embd \\ 0 & crbd \end{bmatrix} \in I$. Assume $ea \neq 0$. Then there exists $n \in M$ such that $0 \neq \begin{bmatrix} ea & em \\ 0 & cr \end{bmatrix} \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ean \\ 0 & 0 \end{bmatrix}$. By above, $0 \neq \left(\begin{bmatrix} ea & em \\ 0 & cr \end{bmatrix} T \right) \cap I$. Assume $cr \neq 0$ but $em = 0$. There exists $d \in R$ such that $0 \neq crd \in y$. Then $0 \neq \begin{bmatrix} ea & em \\ 0 & cr \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & crd \end{bmatrix} \in I$. Since $\begin{bmatrix} e & 0 \\ 0 & c \end{bmatrix}^2 = \begin{bmatrix} e & 0 \\ 0 & c \end{bmatrix}$, T is right FI-extending. □

Corollary 1.6. ([2]) *A ring R is right extending if and only if* $\begin{bmatrix} R & R \\ 0 & R \end{bmatrix}$ *is right extending.*

2. Quasi-extending modules

Let M be a module. A submodule X of M has the quasi-extending property if there exists a direct summand D of M such that : (i) $X \leq D$; (ii) if $K \lhd M$ with $K \cap D \neq 0$, then $K \cap X \neq 0$ (equivalently, if $0 \neq Y \leq D$, then $< Y > \cap X \neq 0$, where $< Y >$ denote the fully in variant submodule of M generated by Y)

We say a module M is quasi-extending if every nonzero submodule X of M has quasi-extending property ([2]).

Proposition 2.1. *Let X be a right ideal of a ring R. If X has the quasi-extending property, then $M_n(X)$ has the quasi-extending property.*

Proof. $M_n(X)$ is a right ideal of $M_n(R)$. Since X has the quasi-extending property, there exists an idempotent $e \in R$ such that $\text{Ret}R \cap X \neq 0$ where $et \neq 0$ and $X \leq eR$.

Let $d = \begin{bmatrix} e & 0 & \cdots & 0 \\ 0 & e & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e \end{bmatrix}$. d is an idempotent of $M_n(R)$.

$M_n(X) \leq dM_n(R)$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \in M_n(R)$ and $dA \neq 0$.

$$dA = \begin{bmatrix} ea_{11} & ea_{12} & \cdots & ea_{1n} \\ ea_{21} & ea_{22} & \cdots & ea_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ea_{n1} & ea_{n2} & \cdots & ea_{nn} \end{bmatrix}$$

We may assume that $ea_{11} \neq 0$. Then $sea_{11} \neq 0$ for some $s \in R$ and $0 \neq sea_{11}r \in X$.

$$\begin{bmatrix} sea_{11}r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in M_n(X) \text{ and}$$

$$\begin{bmatrix} sea_{11}r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in M_n(R)dAM_n(R).$$

$M_n(X) \cap M_n(R)dAM_n(R) \neq 0.$ □

PROPOSITION 2.2. *Let M be a quasi-extending module. If X is a fully invariant submodule of M, then X is a quasi-extending module.*

Proof. Let $A \leq X$. There exists a direct summand D of M such that $A \leq D$ and if $K \lhd M$ with $K \cap D \neq 0$, then $K \cap A \neq 0$. Let $\pi : M \to D$ be the projection endomorphism. $\pi(X)$ is a direct summand of X. $A \leq \pi(X)$. Let $K \lhd X$. Then $K \lhd M$. $K \cap \pi(X) \neq 0$ implies $K \cap D \neq 0$ and hence $K \cap A \neq 0$. □

PROPOSITION 2.3. *If the module $A = B \oplus C$ is a quasi-extending module and B is a fully invariant direct summand of A, then both B and C are quasi-extending modules.*

Proof. By Proposition 2.2, B is a quasi-extending module. To conclude that C is a quasi-extending module, pick a nonzero submodule F of C. Since A is a quasi-extending module, there exists a direct summand D of M such that (i) $F \leq D$; (ii) if $K \lhd M$ with $K \cap D \neq 0$, then $K \cap F \neq 0$.

Since $B \lhd M$ and $B \cap F = 0$, $B \cap D = 0$. $D \cap C$ is a direct summand of C. Consequently, there exists a direct summand D of C such that $F \leq D \leq C$. Let K be a fully invariant submodule of C. Then $B \oplus K$ is a fully invariant submodule of A. Let $K \cap D \neq 0$. Then

$(B \oplus K) \cap D \neq 0$. $(B \oplus K) \cap F \neq 0$. This implies that $K \cap F \neq 0$. Thus C is a quasi-extending module. □

Following simple observation is well-known and we state it for easy reference.

LEMMA 2.4. *Injective modules are quasi-extending modules.*

PROPOSITION 2.5. *Let $M = \oplus_{i \in I} X_i$. Each X_i is a quasi-extending module. Then $A = \oplus_{i \in I} A_i$ has the quasi-extending property where $A_i \leq X_i$ for each $i \in I$.*

Proof. Assume each X_i is a quasi-extending module. A is a submodule of M. Since each X_i is a quasi-extending module, there exists a direct summand D_i of X_i such that (i) $A_i \leq D_i$; (ii) if $K_i \lhd X_i$ with $K_i \cap D_i \neq 0$, then $K_i \cap A_i \neq 0$.

Let $D = \oplus_{i \in I} D_i$. D is a direct summand of M. Let $K \lhd M$ with $K \cap D \neq 0$. $K = \oplus_{i \in I} K_i$ where $K_i = K \cap X_i$. Since K_i is a fully invariant submodule of X_i, $K_i \cap D_i \neq 0$ implies $K_i \cap A_i \neq 0$. Thus $A \cap D \neq 0$. Thus A has the quasi-extending property. □

PROPOSITION 2.6. *Let M be an $H-R$-bimodule where $H = End_R(M)$. Let $T = \begin{bmatrix} H & M \\ 0 & R \end{bmatrix}$ be the formal triangular matrix ring. K is an $H-R$-submodule of M such that $XM \leq K$ where $X \leq H$. If K has the quasi-extending property in M_R and Y has the quasi-extending property in R_R, then $\begin{bmatrix} X & K \\ 0 & Y \end{bmatrix}$ has the quasi-extending property.*

Proof. Let $X \leq H$, $Y \leq K$ and K be an $H-R$-submodule of M such that $XM \leq K$. Assume that K has the quasi-extending property in M_R and Y has the quasi-extending property in R_R. Let $I = \begin{bmatrix} X & K \\ 0 & Y \end{bmatrix} \leq T$. There exists $e = e^2 \in H$ such that $K \leq eM$ and if $A \lhd M$ with $A \cap eM \neq 0$, then $A \cap K \neq 0$. Also there exists an idempotent $c = c^2 \in R$ such that (i) $Y \leq cR$; if $A \lhd R$ with $A \cap cR \neq 0$, then $A \cap Y \neq 0$.

$\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ is an idempotent of T. Since $ea = a$ where $a \in X$, $\begin{bmatrix} X & K \\ 0 & Y \end{bmatrix} \leq \begin{bmatrix} e & 0 \\ 0 & c \end{bmatrix} T$. Let $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ be an ideal of T where $A \lhd H, B \lhd R$ and C is an $H - R$-submodule of M such that $MB \leq C$ and $AM \leq C$.

Let $J = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$. Assume that $\begin{bmatrix} e & 0 \\ 0 & c \end{bmatrix} T \cap J \neq 0$.

$0 \neq \begin{bmatrix} e & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} h & m \\ 0 & r \end{bmatrix} \in J$ and $\begin{bmatrix} eh & em \\ 0 & cr \end{bmatrix} \in J$. If $em \neq 0$, then $em \in C$. $eM \cap C \neq 0$ implies $C \cap K \neq 0$. Thus $J \cap I \neq 0$.

If $eh \neq 0$, $0 \neq \begin{bmatrix} eh & em \\ 0 & cr \end{bmatrix} \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ehn \\ 0 & 0 \end{bmatrix}$, where $ehn \neq 0$ $\quad ehn \in C$. $eM \cap C \neq 0$ implies $C \cap K \neq 0$. Thus $J \cap I \neq 0$.

If $cr \neq 0$, then $B \cap cR \neq 0$. Thus $B \cap Y \neq 0$ and $I \cap J \neq 0$. I has the quasi-extending property. □

THEOREM 2.7. *Let* $T = \begin{bmatrix} S & M \\ 0 & R \end{bmatrix}$ *be a formal triangular matrix ring. If* T *is a right quasi-extending ring, then* R *is a right quasi-extending ring.*

Proof. Let A be a nonzero right ideal of R. Then $\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$ is a right ideal of T. Since T is right quasi-extending, there exists an idempotent $a = \begin{bmatrix} e & m \\ 0 & c \end{bmatrix} \in T$ such that $\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \leq aT$. Let I be an ideal of T such that $I \cap aT \neq 0$. Then $I \cap \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \neq 0$.

$a^2 = \begin{bmatrix} e & m \\ 0 & c \end{bmatrix} \begin{bmatrix} e & m \\ 0 & c \end{bmatrix} = \begin{bmatrix} e^2 & em + mc \\ 0 & c^2 \end{bmatrix} = \begin{bmatrix} e & m \\ 0 & c \end{bmatrix}$.

$e^2 = e$ and $c^2 = c$. Thus e is an idempotent of S and c is an idempotent of R. Since $\begin{bmatrix} e & m \\ 0 & c \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$, $\quad \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \in aT$.

Since $T\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} T \cap \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} = 0$, $e = 0$. $a = \begin{bmatrix} 0 & m \\ 0 & c \end{bmatrix}$ where $mc = m$.

$\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \leq \begin{bmatrix} 0 & m \\ 0 & c \end{bmatrix} T$ implies that $A \leq cR$.

$\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \leq \begin{bmatrix} 0 & m \\ 0 & c \end{bmatrix} T = \left\{ \begin{bmatrix} 0 & mcr \\ 0 & cr \end{bmatrix} \;\middle|\; r \in R \right\}$.

Let $mcr \neq 0$. Then $cr \neq 0$. Let $cr \neq 0$. $T\begin{bmatrix} 0 & mcr \\ 0 & cr \end{bmatrix} T = \begin{bmatrix} 0 & K \\ 0 & RcrR \end{bmatrix}$. $\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \cap T \begin{bmatrix} 0 & mcr \\ 0 & cr \end{bmatrix} T \neq 0$.

Therefore $RcrR \cap A \neq 0$. R is a right quasi-extending ring. □

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