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UNIVALENT HARMONIC EXTERIOR MAPPINGS

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Abstract. In this paper, we will show that the bounds for coefficients of harmonic, orientation-preserving, univalent mappings f defined on $\Delta = \{z : |z| > 1\}$ with $f(\Delta) = \Delta$ are sharp by finding extremal functions.

1. Introduction

Consider the class Σ of all complex-valued, harmonic, orientation-preserving, univalent mappings f defined on $\Delta = \{z : |z| > 1\}$, which are normalized at infinity by $f(\infty) = \infty$ ([1]). If $f \in \Sigma$ with $f(\Delta) = \Delta$, then f has a Poisson integral representation

$$f(z) = \alpha z + \overline{\beta z} + A \log|z| - \frac{1}{2\pi}\int_0^{2\pi} Re\left[\frac{e^{it}+z}{e^{it}-z}\right](e^{i\theta(t)} - \alpha e^{it} - \bar{\beta} e^{-it})dt$$

where θ is a nondecreasing continuous function on $\mathbb{R}$ with $\theta(t+2\pi) = \theta(t) + 2\pi$, $0 \le |\beta| < |\alpha|$, and $|A|/2 \le |\alpha| + |\beta|$ ([1, 2]). Since $Re\left[\frac{e^{it}+z}{e^{it}-z}\right] = -1 - \sum_{m=1}^{\infty}\left(\frac{e^{imt}}{z^m} + \frac{e^{-imt}}{\bar{z}^m}\right)$, we have

$$f(z) = \alpha z + \overline{\beta z} + A \log|z| + \sum_{m=0}^{\infty} \frac{a_m}{z^m} + \sum_{m=1}^{\infty} \frac{\bar{b}_m}{\bar{z}^m}, \tag{1}$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta(t)}dt,$$

$$a_1 = -\bar{\beta} + \frac{1}{2\pi}\int_0^{2\pi} e^{i[t+\theta(t)]}dt, \quad \bar{b}_1 = -\alpha + \frac{1}{2\pi}\int_0^{2\pi} e^{-i[t-\theta(t)]}dt,$$

$$a_m = \frac{1}{2\pi}\int_0^{2\pi} e^{i[mt+\theta(t)]}dt \quad \text{and} \quad \bar{b}_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-i[mt-\theta(t)]}dt$$

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for $m \geq 2$. Thus we know that $|a_m| \leq \frac{1}{m}$, $|b_m| \leq \frac{1}{m}$ for $m \geq 2$ ([2, Corollary 2.5]).

In this paper, we will show that the bounds $|b_m| \leq \frac{1}{m}$ are sharp for $m \geq 2$ by finding an extremal function for each m.

2. Extremal Functions

From now on $n \geq 2$ unless there is further mention. Consider the function $e^{i\theta_n(t)}$, where $\theta_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq 2\pi/n \\ 2\pi & \text{if } 2\pi/n \leq t \leq 2\pi. \end{cases}$ Then $f(z)$ of the form (1) is a harmonic function in Δ and $f(\infty) = \infty$. And also we have

$$a_0 = 1 - \frac{1}{n}, \quad a_m = \begin{cases} -\bar{\beta} - \frac{in(1-e^{i2\pi/n})}{2\pi(n+1)} & \text{if } m = 1 \\ -\frac{in(1-e^{i2\pi m/n})}{2\pi m(n+m)} & \text{if } m \geq 2, \end{cases}$$

and

$$\bar{b}_m = \begin{cases} -\alpha + \frac{in(1-e^{-i2\pi/n})}{2\pi(n-1)} & \text{if } m = 1 \\ \frac{1}{n} & \text{if } m = n \\ \frac{in(1-e^{-i2\pi m/n})}{2\pi m(n-m)} & \text{if } m \neq n,\ m \geq 2. \end{cases}$$

Thus we obtain

$$\begin{aligned} (2) \quad f(z) &= \alpha z + \overline{\beta z} + A \log|z| + (1 - \frac{1}{n}) - \frac{\bar{\beta}}{z} - \frac{\alpha}{\bar{z}} \\ &\quad - \frac{in}{2\pi} \sum_{m=1}^{\infty} \frac{1 - e^{i2\pi m/n}}{m(n+m)z^m} + \frac{in}{2\pi} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{1 - e^{-i2\pi m/n}}{m(n-m)\bar{z}^m} + \frac{1}{n\bar{z}^n} \end{aligned}$$

and $b_n = \frac{1}{n}$.

In the remainder of this section we shall show that there exists a choice of α, β, and A so that f is univalent, hence an extremal function in Σ.

It is obvious that $\sum_{j=1}^{\infty} \frac{\zeta^j}{j} = -\log(1-\zeta)$ for $|\zeta| < 1$. By replacing ζ by $1/z$ and $e^{i2\pi/n}/z$, we obtain

$$(3) \qquad \sum_{j=1}^{\infty} \frac{1 - e^{i2\pi j/n}}{jz^j} = \log \frac{z - e^{i2\pi/n}}{z-1} = L(z)$$

for $|z| > 1$. From (3), we have

(4)
$$\begin{aligned}\sum_{m=1}^{\infty}\frac{1-e^{i2\pi m/n}}{m(n+m)z^m} &= \frac{1}{n}\left[L(z)-\sum_{m=1}^{\infty}\frac{1-e^{i2\pi m/n}}{(n+m)z^m}\right]\\ &= \frac{1}{n}\left[(1-z^n)L(z)+z^n\sum_{j=1}^{n}\frac{1-e^{i2\pi j/n}}{jz^j}\right]\end{aligned}$$

by letting $j = n + m$ and then by applying (3) again. Similarly we also have

$$\sum_{\substack{m=1\\ m\neq n}}^{\infty}\frac{1-e^{i2\pi m/n}}{m(n-m)z^m} = \frac{1}{n}\left[(1-\frac{1}{z^n})L(z)+\frac{1}{z^n}\sum_{j=1}^{n-1}\frac{1-e^{-i2\pi j/n}}{jz^{-j}}\right]. \tag{5}$$

Substitution of (4) and (5) into (2) leads to the representation

$$\begin{aligned}f(z) =&\alpha z+\overline{\beta z}+A\log|z|+(1-\frac{1}{n})-\frac{\bar\beta}{z}-\frac{\alpha}{\bar z}+\frac{1}{n\bar z^n}\\ &-\frac{i}{2\pi}\left[(1-z^n)L(z)+z^n\sum_{j=1}^{n}\frac{1-e^{i2\pi j/n}}{jz^j}\right]\\ &+\frac{i}{2\pi}\overline{\left[(1-\frac{1}{z^n})L(z)+\frac{1}{z^n}\sum_{j=1}^{n-1}\frac{1-e^{-i2\pi j/n}}{jz^{-j}}\right]}.\end{aligned} \tag{6}$$

Thus

(7)
$$\begin{cases} f_z = \alpha+\frac{A}{2z}+\frac{\bar\beta}{z^2}+\frac{inz^{n-1}}{2\pi}\left[L(z)-\sum_{j=1}^{n}\frac{1-e^{i2\pi j/n}}{jz^j}\right], \\ \overline{f_{\bar z}} = \beta+\frac{\bar A}{2z}+\frac{\bar\alpha}{z^2}+\frac{in}{2\pi z^{n+1}}\left[\sum_{j=1}^{n-1}\frac{1-e^{-i2\pi j/n}}{jz^{-j}}-L(z)\right]-\frac{1}{z^{n+1}}. \end{cases}$$

Let $\xi_n = \begin{cases}\frac{4\cos(\pi/n)-1}{6\cos(\pi/n)} & \text{for } n\geq 3\\ 0 & \text{for } n=2.\end{cases}$ Then $\frac{1}{3}\leq \xi_n < \frac{1}{2}$ for $n \geq 3$ and Lemma 2.1 is true for this ξ_n. We made this choice for ξ_n with help of a computer. If we choose $\alpha = ke^{i(n-1)\pi/n}$, $\beta = k\xi_n e^{-i\pi/n}$, and $A = k$ in (6), then the necessary conditions $0 \leq |\beta| < |\alpha|$ and $|A|/2 \leq |\alpha| + |\beta|$ for the mapping in Σ are satisfied for all $k > 0$. Therefore, from now on we restrict α, β, and A to be of this form. In order to prove our main theorem, we need several lemmas. In Theorem 2.7, we are going to show for each $n \geq 2$ that there exist α, β, and A of the form above such that $f \in \Sigma$ with $b_n = 1/n$.

LEMMA 2.1. *Let* $Y = 2\cos[(n-1)\pi/n - \theta(n-1)] + \cos n\theta + 2\xi_n \cos[\pi/n - \theta(n+1)]$ *for* $0 \leq \theta \leq 2\pi/n$. *Then* $Y > 0$.

Proof. If $n = 2$, then $Y \geq 1$ since $0 \leq \sin\theta \leq 1$. In case $n \geq 3$, let $n\theta = t$. Then $0 \leq t \leq 2\pi$ and

$$Y = 2\cos[\pi - (\pi - t)/n - t] + \cos t + 2\xi_n \cos[(\pi - t)/n - t]. \tag{8}$$

Since $Y(2\pi - t) = Y(t)$, we only need to show $Y > 0$ for $0 \leq t \leq \pi$. We can rewrite (8) as follows

$$Y = 2\sin\frac{(2n-1)t+\pi}{2n}\sin\frac{\pi - t}{2n} + 4\xi_n \sin\frac{\pi - t}{n}\sin t + (2\xi_n - 1)\cos\frac{\pi + (n-1)t}{n}$$

by using properties of trigonometric functions.

If $\frac{(n-2)\pi}{2(n-1)} < t \leq \pi$, then $\cos\frac{\pi+(n-1)t}{n} < 0$, $\sin\frac{(2n-1)t+\pi}{2n}\sin\frac{\pi-t}{2n} \geq 0$, and $\sin\frac{\pi-t}{n}\sin t \geq 0$. Thus $Y > 0$ because $1/3 \leq \xi_n \leq 1/2$ for $n \geq 3$.

If $0 \leq t \leq \frac{(n-2)\pi}{2(n-1)}$, then $0 \leq \cos\frac{\pi+(n-1)t}{n} \leq \cos\frac{\pi}{n}$, $\sin\frac{\pi}{2n} \leq \sin\frac{(2n-1)t+\pi}{2n}$, $\sin\frac{\pi}{4(n-1)} \leq \sin\frac{\pi-t}{2n}$, and $\sin\frac{\pi-t}{n}\sin t \geq 0$. Thus $Y \geq 2\sin\frac{\pi}{2n}[\sin\frac{\pi}{4(n-1)} - \frac{1}{3}\sin\frac{\pi}{2n}]$ because $\xi_n = \frac{4\cos(\pi/n)-1}{6\cos(\pi/n)}$ for $n \geq 3$. Since $\sin\frac{\pi}{2n} > 0$ for $n \geq 3$ and $\sin\frac{\pi}{4(n-1)} - \frac{1}{3}\sin\frac{\pi}{2n} \geq \sin\frac{\pi}{4(n-1)} - \frac{\pi}{6n}$, it is enough to show that $\sin\frac{\pi}{4(n-1)} - \frac{\pi}{6n}$ is positive. Let $x = \frac{\pi}{4(n-1)}$, then $0 < x \leq \frac{\pi}{8}$ and $\sin\frac{\pi}{4(n-1)} - \frac{\pi}{6n} = \sin x - \frac{2\pi x}{3(\pi+4x)}$, say $g(x)$. Since $g(0) = 0$, $g(\frac{\pi}{8}) > 0$, and $g'(x) > 0$, we have $g(x) > 0$ for $0 < x \leq \frac{\pi}{8}$. Therefore $\sin\frac{\pi}{4(n-1)} - \frac{\pi}{6n} > 0$ for $n \geq 3$. This implies that $Y > 0$ for $0 \leq t \leq \frac{(n-2)\pi}{2(n-1)}$. □

LEMMA 2.2. *Let* $\zeta = H(z) = \frac{z-e^{i2\pi/n}}{z-1}$ *for* $|z| \geq 1$. *Then the image of* $|z| \geq 1$ *is the half plane* $\pi/n - \pi \leq \arg\zeta \leq \pi/n$ *and* $\arg H(e^{i\theta}) = \begin{cases} \pi/n & \text{if } 2\pi/n < \theta < 2\pi \\ \pi/n - \pi & \text{if } 0 < \theta < 2\pi/n \ . \end{cases}$

Proof. The proof is elementary. We omit it. □

LEMMA 2.3. *Let* $L(z) = \log \frac{z-e^{i2\pi/n}}{z-1} = a(z) + ib(z)$, *and* $z = re^{i\theta}$ $(r \geq 1)$.

$$\text{Then} \begin{cases} a(re^{i\theta}) < 0 & \text{if } \pi/n < \theta < \pi + \pi/n \\ a(re^{i\theta}) = 0 & \text{if } \theta = \pi/n,\ \pi + \pi/n \\ a(re^{i\theta}) > 0 & \text{if } 0 \leq \theta < \pi/n,\ \pi + \pi/n < \theta \leq 2\pi. \end{cases}$$

Proof. In a straight-forward manner, one analyzes the situation when $\left|\frac{z-e^{i2\pi/n}}{z-1}\right|$ is less than 1, equal to 1, and larger than one. □

In the following lemma, we compare derivatives on $|z| = 1$.

LEMMA 2.4. *Let* $z = e^{i\theta}(0 \leq \theta < 2\pi)$. *Then function* (6) *satisfies* $|f_z| = |f_{\bar{z}}|$ *for* $2\pi/n < \theta < 2\pi$, *and if* $0 < \theta < 2\pi/n$, *then* $|f_{\bar{z}}| < |f_z|$ *for all* k *sufficiently large.*

Proof. From (7), we have

$$|f_z| = \frac{n}{2\pi}\left|\frac{2\pi}{inz^{n-1}}(\alpha + \frac{A}{2z} + \frac{\bar{\beta}}{z^2}) + L(z) - \sum_{j=1}^{n} \frac{1-e^{i2\pi j/n}}{jz^j}\right|,$$

$$|f_{\bar{z}}| = \frac{n}{2\pi}\left|\frac{2\pi z^{n+1}}{in}(\beta + \frac{\bar{A}}{2z} + \frac{\bar{\alpha}}{z^2} - \frac{1}{z^{n+1}} + \sum_{j=1}^{n-1} \frac{z^j(1-e^{-i2\pi j/n})}{j} - L(z)\right|.$$

Since $z\bar{z} = 1$, we have

$$\begin{aligned} |f_{\bar{z}}| =& \frac{n}{2\pi}\left|\frac{2\pi}{in\bar{z}^{n+1}}(\beta + \frac{\overline{Az}}{2} + \overline{\alpha z^2} - \bar{z}^{n+1}) + \sum_{j=1}^{n-1} \frac{1-e^{-i2\pi j/n}}{j\bar{z}^j} - L(z)\right| \\ =& \frac{n}{2\pi}\left|\frac{2\pi}{inz^{n-1}}(\frac{\bar{\beta}}{z^2} + \frac{A}{2z} + \alpha - z^{n-1}) - \sum_{j=1}^{n-1} \frac{1-e^{i2\pi j/n}}{jz^j} + \overline{L(z)}\right|. \end{aligned}$$

Let $\frac{2\pi}{inz^{n-1}}(\alpha+\frac{A}{2z}+\frac{\bar{\beta}}{z^2})-\sum_{j=1}^{n}\frac{1-e^{i2\pi j/n}}{jz^j}=K(z)=c(z)+id(z)$. Then

$$\begin{aligned}|f_z| =&\frac{n}{2\pi}|K+L| = \frac{n}{2\pi}|(c+a)+i(d+b)| \text{ and}\\ |f_{\bar{z}}| =&\frac{n}{2\pi}\left|K+\bar{L}-\frac{2\pi}{in}\right| = \frac{n}{2\pi}\left|(c+a)+i(d+\frac{2\pi}{n}-b)\right|.\end{aligned}$$

From these, we have $|f_z|^2-|f_{\bar{z}}|^2=\frac{n^2}{\pi^2}(d+\frac{\pi}{n})(b-\frac{\pi}{n})$. By Lemma 2.2, we know that

$$b(e^{i\theta})=\begin{cases}\pi/n & \text{if } 2\pi/n<\theta<2\pi\\ \pi/n-\pi & \text{if } 0<\theta<2\pi/n.\end{cases}$$

Therefore,

$$|f_z|^2-|f_{\bar{z}}|^2=\begin{cases}0 & \text{if } \frac{2\pi}{n}<\theta<2\pi\\ -\frac{n^2}{\pi}(d+\frac{\pi}{n}) & \text{if } 0<\theta<\frac{2\pi}{n}.\end{cases}\tag{9}$$

Now for $0<\theta<2\pi/n$, we want to show that $|f_{\bar{z}}|<|f_z|$ for all k sufficiently large. Since $\alpha=ke^{i(n-1)\pi/n}$, $\beta=k\xi_n e^{-i\pi/n}$, and $A=k$, we have

$$\begin{aligned}d=-\frac{\pi k}{n}&\{2\cos(n-1)(\pi/n-\theta)+\cos n\theta+2\xi_n\cos[\pi/n-(n+1)\theta]\}\\ &+\sum_{j=1}^{n-1}\frac{\sin\theta j+\sin(\frac{2\pi j}{n}-\theta j)}{j}.\end{aligned}$$

By using the notation of Lemma 2.1, we have

$$d=-\frac{\pi k}{n}Y+\sum_{j=1}^{n-1}\frac{\sin\theta j+\sin(2\pi/n-\theta)j}{j}.$$

By substituting d into (9), we have

$$|f_z|^2-|f_{\bar{z}}|^2=nkY-\frac{n^2}{\pi}\sum_{j=1}^{n-1}\frac{\sin\theta j+\sin(2\pi/n-\theta)j}{j}-n.$$

From Lemma 2.1, we know that $C = min_{0\leq\theta\leq 2\pi/n} Y > 0$. Thus for $0 < \theta < 2\pi/n$,

$$|f_z|^2 - |f_{\bar{z}}|^2 \geq n\left[kC - \frac{n}{\pi}\sum_{j=1}^{n-1}\frac{\sin\theta j + \sin(2\pi/n-\theta)j}{j} - 1\right].$$

Since $\frac{n}{\pi}\sum_{j=1}^{n-1}\frac{\sin\theta j+\sin(2\pi/n-\theta)j}{j} + 1$ is bounded, there exists $M_n > 0$ such that $\left|\frac{n}{\pi}\sum_{j=1}^{n-1}\frac{\sin\theta j+\sin(2\pi/n-\theta)j}{j} + 1\right| \leq M_n$. Therefore $|f_z|^2 - |f_{\bar{z}}|^2 \geq n(kC - M_n)$. Choose k such that $k > M_n/C$. Then $|f_z|^2 - |f_{\bar{z}}|^2 > 0$, that is, $|f_z| > |f_{\bar{z}}|$. □

LEMMA 2.5. *On* $E = \{z : |z| \geq 1, z \neq 1, z \neq e^{i2\pi/n}\}$, *we have* $f_z \neq 0$ *for all* k *sufficiently large.*

Proof. Let $F = k(-e^{-i\pi/n} + \frac{1}{2z} + \frac{\xi_n e^{i\pi/n}}{z^2})$, and let
$S = \frac{inz^{n-1}}{2\pi}\sum_{j=n+1}^{\infty}\frac{1-e^{i2\pi j/n}}{jz^j}$. Then $f_z = F + S$ by applying (3).

Part I: Let $z = re^{i\theta}$. Then we have $|S| \leq \frac{nr^{n-1}}{\pi}\sum_{j=n+1}^{\infty}\frac{1}{jr^j} < \frac{n}{(n+1)\pi r(r-1)}$ for $r > 1$ and

$$|F| \geq k(1 - \frac{1}{2r} - \frac{\xi_n}{r^2}) \geq k(\frac{1}{2} - \xi_n) \text{ for } r \geq 1. \tag{10}$$

Thus $|f_z| > k(\frac{1}{2} - \xi_n) - \frac{n}{(n+1)\pi r(r-1)}$ for $r > 1$. If $r \geq 2$, then $|f_z| > 0$ for all $k \geq \frac{n}{(n+1)\pi(1-2\xi_n)}$.

Part II: Let $\tilde{F}(z) = \frac{F}{k}$, $\Theta_1 = \{w : Re\ w > 0,\ Im\ w > 0\}$, $\Theta_2 = \{w : Re\ w < 0,\ Im\ w > 0\}$, $\Theta_3 = \{w : Re\ w < 0,\ Im\ w < 0\}$, and $\Theta_4 = \{w : Re\ w > 0,\ Im\ w < 0\}$. Then $\tilde{F}(1) \in \Theta_1$, $\tilde{F}(e^{i2\pi/n}) \in \Theta_2$ for $n = 2, 3$, and $\tilde{F}(e^{i2\pi/n}) \in \Theta_3$ for $n \geq 4$ since $\xi_n = \begin{cases} \frac{4\cos(\pi/n)-1}{6\cos(\pi/n)} & \text{if } n \geq 3 \\ 0 & \text{if } n = 2 \end{cases}$.

Choose ϵ $(0 < \epsilon < \frac{\pi}{2n})$ such that $\{w : |\tilde{F}(1) - w| < 2\epsilon\} \subset \Theta_1$ and $\{w : |\tilde{F}(e^{i2\pi/n}) - w| < 2\epsilon\} \subset \Theta_2$ for $n = 2, 3$, and $\{w : |\tilde{F}(e^{i2\pi/n}) - w| < 2\epsilon\} \subset \Theta_3$ for $n \geq 4$. Since $\tilde{F}$ is continuous at $z = 1,\ e^{i2\pi/n}$, there exists $\delta > 0$ for which $|\tilde{F}(z) - \tilde{F}(1)| < \epsilon$ if $|z - 1| < \delta$ and $|\tilde{F}(z) - \tilde{F}(e^{i2\pi/n})| < \epsilon$ if $|z - e^{i2\pi/n}| < \delta$, and $\{z : |z-1| < \delta\} \cap \{z : |z - e^{i2\pi/n}| < \delta\} = \phi$. Thus we have the following facts:

(11) *i*) For $|z - 1| < \delta$, we have $Re\tilde{F} > \epsilon/2$ and $Im\tilde{F} > \epsilon/2$.

(12)
$ii)$ If $n = 2, 3,$ then $Re\tilde{F} < -\epsilon/2$ and $Im\tilde{F} > \epsilon/2$ for $|z - e^{i2\pi/n}| < \delta$.

(13)
$iii)$ If $n \geq 4,$ then $Re\tilde{F} < -\epsilon/2$ and $Im\tilde{F} < -\epsilon/2$ for $|z - e^{i2\pi/n}| < \delta$.

Now consider $L(z) = \log \frac{z - e^{i2\pi/n}}{z-1}$. Then by Lemmas 2.2 and 2.3, we know that $\lim_{z\to 1} a(z) = \infty,\ \lim_{z\to e^{i2\pi/n}} a(z) = -\infty$, and $L(|z| \geq 1) = \{w : \pi/n - \pi \leq\ Im\ w\ \leq \pi/n\}$. For $M = \frac{(n-1)\pi}{n \tan \epsilon},\ |z| \geq 1$, there is a $\delta_1 (0 < \delta_1 \leq \delta)$ for which $a(z) > M$ if $|z - 1| < \delta_1$ and $a(z) < -M$ if $|z - e^{i2\pi/n}| < \delta_1$. Choose δ_0 small enough that

$i)$ $\delta_0 < \delta_1$,

$ii)$ $z = re^{i\theta} \in \{z : |z - 1| < \delta_0,\ |z| \geq 1\} = Q_1 \Rightarrow 0 \leq \theta < \dfrac{\pi}{2n}$ or $2\pi - \dfrac{\pi}{2n} < \theta \leq 2\pi$,

$iii)$ $z = re^{i\theta} \in \{z : |z - e^{i2\pi/n}| < \delta_0,\ |z| \geq 1\} = Q_2 \ \Rightarrow\ \dfrac{3\pi}{2n} < \theta < \dfrac{5\pi}{2n}$.

Then $\left|\arg\left[\frac{inL(z)}{2\pi}\right] - \frac{\pi}{2}\right| < \epsilon$ for $z \in Q_1,$ and $\left|\arg\ \left[\frac{inL(z)}{2\pi}\right] - \frac{3\pi}{2}\right| < \epsilon$ for $z \in Q_2$. This implies that

$$\frac{inz^{n-1}}{2\pi} L(z) \in \{w : Im\ w > 0\} \quad \text{for } z \in Q_1 \tag{14}$$

and for $z \in Q_2$,

$$\frac{inz^{n-1}}{2\pi} L(z) \notin \overline{\Theta}_3 \cup \overline{\Theta}_4 \quad \text{if } n = 2, \tag{15}$$

$$\frac{inz^{n-1}}{2\pi} L(z) \notin \overline{\Theta}_4 \quad \text{if } n = 3, \tag{16}$$

$$\frac{inz^{n-1}}{2\pi} L(z) \notin \overline{\Theta}_1 \quad \text{if } n \geq 4. \tag{17}$$

The proof above is elementary. So we omit it.

By substituting $L(z)$ and $\tilde{F}(z)$ into (7), we have

$$f_z = k\tilde{F} + \frac{inz^{n-1}}{2\pi}L - \frac{inz^{n-1}}{2\pi}\sum_{j=1}^{n}\frac{1-e^{i2\pi j/n}}{jz^j}.$$

Since $\frac{-inz^{n-1}}{2\pi}\sum_{j=1}^{n}\frac{1-e^{i2\pi j/n}}{jz^j}$ is a polynomial of degree $n-2$, there exists $M_0 > 0$ such that $\left|-\frac{inz^{n-1}}{2\pi}\sum_{j=1}^{n}\frac{1-e^{i2\pi j/n}}{jz^j}\right| \leq M_0$ on $\Theta = Q_1 \cup Q_2$. So on Θ, We have

$$|f_z| \geq \left|k\tilde{F} + \frac{inz^{n-1}}{2\pi}L\right| - M_0. \tag{18}$$

If $z \in Q_1$, then from (11) and (14), we obtain

$$|f_z| \geq \left|k\tilde{F} + \frac{inz^{n-1}}{2\pi}L\right| - M_0 \geq Im\left[k\tilde{F} + \frac{inz^{n-1}}{2\pi}L\right] - M_0 > \frac{k\epsilon}{2} - M_0. \tag{19}$$

Similarly, when $z \in Q_2$, apply (12) and (15), (12) and (16), (13) and (17) to (18), then we also obtain

$$|f_z| > \frac{k\epsilon}{2} - M_0. \tag{20}$$

From (19) and (20), we have $|f_z| > k\epsilon/2 - M_0$ on Θ. Therefore $f_z \neq 0$ on Θ for $k \geq 2M_0/\epsilon$.

Part III: Let $\bar{\text{U}} = \{z : 1 \leq |z| \leq 2\}\backslash\Theta$. Then there exists $M_1 > 0$ such that $|L(z)| \leq M_1$. By applying (3) to S, we have

$$S = \frac{inz^{n-1}}{2\pi}\left(L(z) - \sum_{j=1}^{n}\frac{1-e^{i2\pi j/n}}{jz^j}\right).$$

Thus

$$|S| \leq \frac{n|z|^{n-1}}{2\pi}\left(M_1 + \sum_{j=1}^{n-1}\frac{2}{j|z|^j}\right) \leq \frac{nM_1 2^{n-2}}{\pi} + \frac{n}{\pi}\sum_{j=1}^{n-1}\frac{2^{n-1-j}}{j} = R \text{ on } \bar{\text{U}}.$$

(10) implies that $|f_z| \geq k(1/2 - \xi_n) - R$ on $\bar{\cup}$. If $k > 2R/(1-2\xi_n)$, then $|f_z| > 0$ on $\bar{\cup}$.

Part I, II, and III imply that for all

$$k > max\left\{\frac{n}{(n+1)\pi(1-2\xi_n)}, \frac{2M_0}{\epsilon}, \frac{2R}{1-2\xi_n}\right\}$$

$f_z \neq 0$ on E. □

LEMMA 2.6. *The function $f(z)$ of the form (6) with $\alpha = ke^{i(n-1)\pi/n}$, $\beta = k\xi_n e^{-i\pi/n}$, and $A = k$ is locally univalent in Δ for all k sufficiently large.*

Proof. By Lemma 2.4, there exists k_1 such that $|f_z| \geq |f_{\bar{z}}|$ on $B = \{z : |z| = 1,\ z \neq 1,\ z \neq e^{i2\pi/n}\}$ for all $k > k_1$. By Lemma 2.5, there exists k_2 such that $f_z \neq 0$ on $E = \{z : |z| \geq 1,\ z \neq 1,\ z \neq e^{i2\pi/n}\}$ for all $k > k_2$. Let $k_f = max\{k_1, k_2\}$. Then for all $k > k_f$, we have $|f_z| \geq |f_{\bar{z}}|$ for $z \in B$ and $f_z \neq 0$ for $z \in E$.

Let $\tilde{a}(z) = \overline{f_{\bar{z}}}/f_z$, then $\tilde{a}(z)$ is analytic on Δ and $|\tilde{a}(z)| \leq 1$ on B.

$$|\tilde{a}(z)| = \left|\frac{\overline{f_{\bar{z}}}}{f_z}\right| = \left|\frac{\text{Regular terms}/L(z) - \frac{in}{2\pi z^{n+1}}}{\text{Regular terms}/L(z) + \frac{inz^{n-1}}{2\pi}}\right| \to 1$$

as $z \to 1$ or $z \to e^{i2\pi/n}$. By the Maximum Principle, we have $|\tilde{a}(z)| \leq 1$ in Δ. If $|\tilde{a}| = 1$ at some point in Δ, then $|\tilde{a}| \equiv 1$; but $|\tilde{a}(\infty)| = |\beta/\alpha| = \xi_n < 1/2$. This implies that $|\tilde{a}| < 1$ in Δ. Thus $|f_z| > |\overline{f_{\bar{z}}}|$ in Δ. Therefore f is locally univalent in Δ, and at ∞, too. □

THEOREM 2.7. *For each $n \geq 2$ there exists a harmonic, orientation-preserving, univalent mapping of Δ onto itself with the Fourier expansion (1) such that $b_n = \frac{1}{n}$.*

Proof. For each n, take (6) with $\alpha = ke^{i(n-1)\pi/n}$, $\beta = k\xi_n e^{-i\pi/n}$, and $A = k$ where $\xi_n = \begin{cases} \frac{4\cos(\pi/n)-1}{6\cos(\pi/n)} & \text{if } n \geq 3 \\ 0 & \text{if } n = 2. \end{cases}$ Then there exists $k > 0$ such that f is local homeomorphism on $\Delta \cup \{\infty\}$ and the Jacobian of f, $J_f = |f_z|^2 - |f_{\bar{z}}|^2$, is positive on $\Gamma = \{z = e^{i\theta} : 0 < \theta < 2\pi/n\}$ by Lemma 2.6 and Lemma 2.4. f is a local homeomorphism in full neighborhood of each point p of Γ to some neighborhood of $f(p)$ since $J_f > 0$ on Γ.

Now define the reflection $G(z) = \begin{cases} f(z) & \text{if } |z| \geq 1 \\ \frac{1}{\overline{f(1/\bar{z})}} & \text{if } |z| < 1. \end{cases}$ Then G is local homeomorphism at each point of $S^2 \backslash \tilde{\Gamma}$ where S^2 is the Riemann sphere and $\tilde{\Gamma} = \{z = e^{i\theta} : 2\pi/n \leq \theta \leq 2\pi\}$. It is continuous on S^2. However, G is constant on $\tilde{\Gamma}$. Now identify points of $\tilde{\Gamma}$ and call this element b. We obtain a new function F on a new domain, which is topologically a sphere S^2. F is a homeomorphism since F is a local homeomorphism on $S^2 \backslash \{b\}$ and continuous on S^2. Hence G is a homeomorphism on $S^2 \backslash \tilde{\Gamma}$. Therefore, $G\Big|_{\Delta} = f$ is a homeomorphism. □

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